ON THE THEORY OF MATRICES WITH ELEMENTS IN THE CLEBSCH-ARONHOLD SYMBOLIC CALCULUS

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On the Theory of Matrices with elements in the Clebsch-Aronhold Symbolic Calculus.

In the theory of invariant matrices and in the classical invariant theory there arise a considerable number of rather surprising isomorphisms relating to symmetric functions of the latent roots and to the theory of matric representations of the symmetric group. Although these relations appear sooner or later in the development of either theory it would seem, on account of their fundamental simplicity, at least desirable that they should be brought into evidence by an analysis of the algebraic nature of the systems involved. Since many of the relations referred to are merely extensions of familiar determinantal theorems, and since the totality of reducing matrices for the general invariant matrix should give a complete basis for determinantal relations, the development of these results should embody an extension of determinant theory besides all results relating to determinants with unrestricted elements in the ground field.

With this end in view, the discussion given below proceeds first from an analysis of the nature of various expressions involving determinants and permanents, to a construction for the orthogonal representations of the symmetric group. The reductions given here are related to the corresponding reductions for the central cores of invariant matrices by means of an isomorphism which can be expressed in terms of certain symbolic quantities obtained by writing each element of the fundamental matrix $[a_{ij}]$ as a symbolic product $a_i \alpha_j$ and forming direct products of compounds and Sohläflians of $[a_{ij}]$ by the use of equivalent symbols. A symbolisation of matrices in this way has been used by Professor Turnbull in his paper "The Invariant Theory of a General Bi-linear Form" (Proc. L.M.S. Series II, Vol. 33, Part I). The process of deduction followed in this paper is, however, essentially non-symbolic. It could have been framed equally well in the terminology of invariant matrices, in which it forms an extension of some theorems developed by Professor Aitken in his Research Lectures. The actual representations obtained are essentially the same as the orthogonal forms developed by Young.

The discussion is restricted to the orthogonal case for the sake of symmetry, although rational forms can be developed in a similar manner.

In the paper quoted above, Professor Turnbull uses his symbolic forms to obtain expressions representing symmetric functions of the latent roots of the matrix $[a_{ij}]$. His results are easily extended to the complete homogeneous symmetric functions and to bi-alternants, and suggest the manner in which the reducing matrices for the central core might be extended to reduce the full invariant matrix. A homo-morphism is developed for this extension /

1. THE LINEAR FORM MODUL AND THE GROUP RING.

Consider first the construction of the most general type of expression which may be regarded as a determinantal form. The essential character of such expressions being their relation to the symmetric group, One might proceed in the following manner.

Let $\mathfrak{M}$ be a linear form modul of rank $n!$ in a field K of characteristic zero. The modul, an additive Abelian group, is determined completely, apart from isomorphism, by K and the number $n!$ Let its basis elements be denoted by $u_1, u_2 \cdots u_{n!}$

The $n!$ basis elements being completely arbitrary, they can be associated with the permutations of the symmetric group $\mathfrak{H}_{[n]}$ of order $n!$ on symbols $1, 2 \cdots n$ by means of a correspondence which can be written

$$S_i = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \longleftrightarrow u_i$$

The quantities could now be identified with their corresponding S_i and multiplication in $\mathfrak{M}$ could be defined by means of the multiplication table of $\mathfrak{H}_{[n]}$. The form would then become a hyper complex system - the group ring $\mathcal{O}$ of the symmetric group over the ground field K.

Instead, one might define right-handed multiplication of the basis elements of $\mathfrak{M}$ by elements of $\mathfrak{H}_{[n]}$ from the group multiplication table

$$s_j \cdot u_i = u_k \qquad\qquad s_j s_i = s_k$$

The elements of $\mathfrak{H}_{[n]}$ then form a left-handed multiplication domain for $\mathfrak{M}$, which may be extended to the group ring $\mathcal{O}$ by defining addition of operators so that

$$(k_1 s_i + k_2 s_j)\, u_1 = s_i \cdot k_1 u_1 + s_j \cdot k_2 u_1 \qquad\qquad k_1, k_2 \in K$$

The distinction between the modul $\mathcal{O}\mathfrak{M}$ and the group ring is essential to the development of the theory, since the ring $\mathcal{O}$ exhibits various properties not possessed by $\mathfrak{M}$. As the theory depends mainly on the different isomorphisms which can be established, it will, in particular, be important to distinguish between the ring-isomorphisms of $\mathcal{O}$ which have their analogues in $\mathfrak{M}$, and the operator-isomorphisms of $\mathcal{O}\cdot\mathfrak{M}$

Both $\mathfrak{M}$ and $\mathcal{O}$ admit the isomorphism

$$a \longleftrightarrow a^* \; ; \qquad m \longleftrightarrow m^* \qquad\qquad a \in \mathcal{O}\, ; \; m \in \mathfrak{M},$$

in which the symbols s_i, u_i in terms of which a and m are expressed remain unaltered or change their signs according as s_i is or is not contained in the alternating group $\mathfrak{A}_n$. The validity of this isomorphism in $\mathcal{O}$ depends on the fact that $\mathfrak{A}_n$ is a normal division of $\mathfrak{H}_{[n]}$, with an Abelian factor group. Since $\mathfrak{A}_n$ is also the commutator sub-group, this is the only automorphism of the type considered. Quantities obtained by applying the automorphism will be indicated by an asterisk and the process referred to as one of taking conjugates with respect to $\mathfrak{A}_n$.

It must be remarked that the conjugate relation is a ring automorphism in $\mathcal{O}$ from which a corresponding automorphism is obtained in the modul, and not an operator automorphism of $\mathcal{O}\cdot\mathfrak{M}$.

The following example illustrates the application of the conjugate relation and develops some notations which will be required later. Let $[\lambda] = [\lambda_1, \cdots \lambda_h]$ denote a partition of n and $\mathfrak{H}_{[\lambda]}$ denote the direct product of the symmetric groups of orders $\lambda_1!, \lambda_2! \ldots\ldots \lambda_h!$ on the first λ_1, the next λ_2, and so on till the last λ_h of the symbols $1, 2 \cdots\cdots n$ respectively. Correspondingly in $\mathcal{O}$ write

$$S_{[\lambda]} = \sum_{\tau \in \mathfrak{H}_{[\lambda]}} \tau$$

If $u = \binom{n}{\lambda}$, the index of $\mathfrak{H}_{[\lambda]}$ in $\mathfrak{H}_{[n]}$, and $\tau_1, \tau_2 \ldots\ldots\ldots \tau_u$ are substitutions belonging to different left co-sets of $\mathfrak{H}_{[\lambda]}$ then, in the notation of group theory,

$$\mathfrak{H}_{[n]} = \tau_1 \mathfrak{H}_{[\lambda]} + \tau_2 \mathfrak{H}_{[\lambda]} + \cdots\cdots + \tau_u \mathfrak{H}_{[\lambda]}.$$

Correspondingly in $\mathcal{O}$

$$S_{[n]} = (\tau_1 + \tau_2 + \cdots\cdots\cdots + \tau_u)\, S_{[\lambda]}$$

and in $\mathcal{O}\mathfrak{M}$, if u_1 corresponds to the identity of the group,

$$S_{[n]}\, u_1 = (\tau_1 + \tau_2 + \cdots\cdots + \tau_u)\, S_{[\lambda]}\, u_1$$

and

$$S^*_{[n]} \cdot u_1 = (\tau_1 + \tau_2 + \cdots\cdots\cdots + \tau_u)^* S^*_{[\lambda]} \cdot u_1$$

If the quantities u_i be chosen as the $n!$ products arising from the n^2 elements of a determinant, then the last two equations express the validity of the Laplace development of a determinant and its analogue for permanents. In fact it requires merely a change of notation to write

$$u_i = a_{i_1 i_2 \cdots\cdots i_n}$$

for the quantity corresponding to s_i, and then to extend this to

$$u_i = a_{1i_1} a_{2i_2} \cdots\cdots a_{ni_n} ,$$

in which form the separable parts may be allowed to commute since their order is already fixed by the earlier set of suffixes. The expressions written above then become

$${}^{+}\left| a_{11}\, a_{22} \cdots\cdots a_{nn} \right|^{+} , \qquad \left| a_{11}\, a_{22} \cdots\cdots a_{nn} \right| ,$$

respectively. This notation is particularly adapted to the representation of the endomorphisms of $\mathfrak{M}$ which are investigated in the next paragraph.

Consider now the endomorphisms-ring of $\mathfrak{M}$, with $\mathcal{O}$ as left multiplication domain, and let Γ be an endomorphism in which $u_1 \to u_j$. If the discussion is restricted to operator-endomorphisms, then

$$\Gamma u_i = \Gamma s_i u_1 = s_i \Gamma u_1 = s_i u_j$$

so that the behaviour of the elements of $\mathfrak{M}$ is determined by that of u_1. Moreover, it is clear that, when sums and products of automorphisms are defined, their additive group has a basis

$$\left\{ \Gamma_1 , \Gamma_2 \cdots\cdots \Gamma_{n!} \right\}$$

consisting of those which transform u_1 into each of the other basis elements of $\mathfrak{M}$

Forming the product of two endomorphisms

$$(\Gamma_i \Gamma_j)\, u_1 = \Gamma_i (\Gamma_j u_1) = \Gamma_i u_j = s_j \cdot \Gamma_i u_1 = (s_j s_i) u_1$$

showing that the endomorphism ring is inversely isomorphic to the group ring $\mathcal{O}$.

The concept of the endomorphism ring is very conveniently exhibited in terms of the extended notation for the elements in the following manner. Firstly let the Γ_j be re-ordered by writing $\bar{s}_j$ for the endomorphism in which $u_1 \to s_j^{-1} \cdot u_1$. Then

$$\overline{s_i s_j} \cdot u_1 = \bar{s}_i \cdot \bar{s}_j u_1 = s_j^{-1} \cdot \bar{s}_i u_1 = s_j^{-1} s_i^{-1} u_1 = (s_i s_j)^{-1} u_1 .$$

Writing as usual

$$s_i = \begin{pmatrix} 1 & 2 & \cdots\cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} , \qquad s_j = \begin{pmatrix} 1 & 2 & \cdots\cdots & n \\ j_1 & j_2 & \cdots & n \end{pmatrix} ,$$

and applying the endomorphisms to

$$u_1 = a_{11}\, a_{22} \cdots\cdots a_{nn} \quad ,$$

in which the symbols are allowed to commute, one finds that

$$\bar{S}_i\, u_1 = a_{i_1 1}\, a_{i_2 2} \cdots\cdots a_{i_n n}$$

since re-arrangement of the factors gives the result of applying to u_1 the permutation

$$\begin{pmatrix} i_1 & i_2 & \cdots\cdots & i_n \\ 1 & 2 & \cdots\cdots & n \end{pmatrix} = S_i^{-1}$$

The fact that the two operations commute is expressed by the relation

$$S_i\, \bar{S}_j\, u_1 = a_{j_1 i_1}\, a_{j_2 i_2} \cdots\cdots a_{j_n i_n} = \bar{S}_j\, u_i$$

It will be convenient to denote the inverse ring by the symbol $\bar{\mathcal{O}}$ and to express the isomorphism between quantities of $\mathcal{O}$ and $\bar{\mathcal{O}}$ by means of a horizontal bar. It is also convenient to write endomorphisms after the modul. The commutative property shown above then assumes an associative form.

An illustration of the manner in which the preceding discussion can be applied to determinantal theory is afforded by Schwein's theorem (see for example Turnbull: "The Theory of Determinants, Matrices and Invariants", second edition, page 336), which, if $[p, q, r]$ is a partition of n, can be expressed in the form

$$\left(S^*_{[1^p, q+r]}\, u_1 \right) \bar{S}^*_{[p+q, 1^r]} = S^*_{[1^p, q+r]} \left(u_1\, \bar{S}^*_{[p+q, 1^r]} \right)$$

and is therefore a consequence of the associative law derived above. A factor $q!$ is to be removed from each member. The correspondence between this and the determinantal form

$$\sum^*_{\{i_1 \cdots i_{p+q}\}} a_{i_1 1} \cdots\cdots a_{i_p p} \left| a_{i_{p+1}, p+1} \cdots\cdots a_{i_{p+q}, p+q}\; a_{p+q+1, p+q+1} \cdots\cdots a_{nn} \right|$$

$$= \sum^*_{\{j_{p+1} \cdots j_n\}} \left| a_{11} \cdots\cdots a_{pp}\, a_{p+1, j_{p+1}} \cdots\cdots a_{p+q, j_{p+q}} \right| a_{p+q+1, j_{p+q+1}} \cdots\cdots a_{n j_n}$$

in which the sums extend over all the suffix sets indicated and a common factor $q!$ is removed, is obvious.

In this sense Schwein's theorem appears as an extension of the /

the Laplace development in which the determinants of the expansions are formed by using both $\mathcal{O}$ and $\overline{\mathcal{O}}$. In the first form the generalisation and the corresponding result for permanents is obvious.

2. REPRESENTATION OF THE GROUP AND THE GROUP RING.

The significance of several of the distinctions made in the preceding paragraph is not made clear until the ideas are applied to the theory of group representations, and, in particular, to the study of ideal homo-morphisms. Conversely, this study summarises in a convenient manner many of the properties of $\mathcal{O}\mathfrak{M}$ and gives complete information on certain types of determinantal relations.

Restricting the discussion, for the sake of simplicity, to matrices, in the same field K as the modul, it is an immediate consequence of the laws of scalar multiplication that every representation of the group yields a corresponding representation of the group ring, and conversely. In fact, if $A_i \longleftrightarrow s_i$, then in the group representation there is a homomorphism

$$s_i \longrightarrow A_i \qquad s_i s_j \longrightarrow A_i A_j$$

and the elements u_i of $\mathfrak{M}$, being quite arbitrary, can be replaced by the A_i , and the modul then extended to a ring exactly as was done for the group ring.

Since if one of the matrices A_i is singular, all must be, and they can obviously be reduced simultaneously to the direct sum of non-singular matrices and a null matrix, the discussion may be restricted throughout to the case of non-singular matrices. Then the unit of $\mathcal{O}$ is represented by the unit matrix and s_i^{-1} by A_i^{-1}

Firstly, in the representation of the group, the correspondence is an isomorphism except in the case where a normal divisor is available. Since $\mathfrak{A}_n$ is the only normal divisor of $\mathfrak{H}_{[n]}$ except when $n = 4$ this is in general the only case where the representation is not faithful. There corresponds, therefore, a representation in which only two distinct matrices are used, one for the members of $\mathfrak{A}_n$ and the other for the odd permutations of $\mathfrak{H}_{[n]}$. The former, including <u>inter alia</u> the identical permutation, is represented by the unit matrix, so that the second matrix can always be brought to diagonal form. Thus there arises only the alternating representation in which

$$\mathfrak{A}_n \longrightarrow 1 \quad , \quad (12)\,\mathfrak{A}_n \longrightarrow -1$$

The only other possible homo-morphism is the trivial one in which the entire group is taken as its own divisor and all permutations are represented by a unit.

The exceptional case $n = 4$ is considered in detail in a later section.

NOTE: The extension of language used here perhaps requires some explanation. $\mathfrak{M}$, being a modul with an operator domain $\mathfrak{o}$, cannot possess an ideal in the ordinary sense of the word. $\mathfrak{a}$ is here, more strictly, an 'invariant sub-group' (zülässige Untermodul in bezug auf $\mathfrak{o}$). But, on account of the relation between group representation and ring representation, no confusion can arise when the terminology for $\mathfrak{o}$ is transferred to $\mathfrak{M}$, and, considering other uses of the word invariant, this has been done throughout the part concerning group representations. The symbols u_1, which form the only essential distinction, are accordingly dropped later in the discussion.

On the other hand the symbolic theory of reduction used later belongs essentially to the modul $\mathfrak{o}.\mathfrak{M}$ and the distinction is again introduced.

It is convenient to mention here another consequence of the properties of $\mathfrak{A}_n$. Since it has an Abelian factor group, one can form the conjugate ring to that of any given representation and so obtain, in general, another representation not equivalent to the original. The relation is reciprocal and the two representations will be termed conjugate. In the case of groups, the conjugate representations can be expressed by the same matrices with the signs of those corresponding to substitutions not in $\mathfrak{A}_n$ reversed. Clearly these are either both reducible or both irreducible. $\mathfrak{A}_n$ being the commutator sub-group, there is no other correspondence of this type.

The questions of homo-morphisms within the ring is at first sight more complicated and its full investigation yields the whole theory of group representations. Corresponding to the normal divisor reduction of a group the existence of ideals in $\mathcal{O}$ exhibits the ring homo-morphisms and corresponding to the derivation of scalar representations from Abelian factor-groups, the existence of two-sided ideals is actually the raison d'être of reducibility.

With the construction of orthogonal matrices in view, let orthogonality be defined in $\mathfrak{M}$ with respect to the fundamental basis $u_1, u_2, \ldots\ldots u_{n!}$. Two quantities of $\mathfrak{M}$ will be termed orthogonal if, expressed in terms of $u_1 \ldots\ldots u_{n!}$, the sum over the products of corresponding co-efficients in K is zero. If b_{u_1} denotes the co-efficient of u_1 in any expression, which follows, after this has been expressed in the fundamental basis, the condition for the orthogonality of two quantities $a \cdot u_1$ and $b \cdot u_1$ is obviously

$$b_{u_1}(a\bar{b}u_1) = b_{u_1}(a \cdot u_1 \cdot b) = 0$$

Then let an ideal $\mathfrak{a}$ of $\mathfrak{M}$ have an orthogonal basis

$$\mathfrak{a} = \{e_1 \cdot u_1, e_2 u_1, \ldots\ldots e_f \cdot u_1\}$$

and let an orthogonal basis be selected for the reciprocal space. This is also an ideal in $\mathfrak{M}$ since, if τ is any substitution $\tau \cdot \mathfrak{a} u_1 = \mathfrak{a} \cdot u_1$ and for any b of the reciprocal set τb is orthogonal to the quantities $\tau e_1 u_1, \ldots\ldots \tau e_f u_1$

It follows from this result that any quantity of the reciprocal set, which we may denote by $\mathfrak{b}$, actually annihilates the quantities of $\bar{\mathfrak{a}}$, that is

$$b \cdot u_1 a = b \cdot \bar{a} u_1 = 0, \quad a \in \mathfrak{a},\ b \in \mathfrak{b},$$

for we have

$$b_{u_i}(b\bar{a}u_1) = b_{u_1}(s_i^{-1} b \bar{a} u_1) = b_{u_1}(b' \bar{a} u_1) = 0.$$

The result obtained here is, when $\mathfrak{a}$ is taken as a two-sided/

sided ideal in $\mathfrak{O}$, equivalent to the theorem of Maschke, or to the proof that reducibility of the group ring implies its complete reducibility.

Suppose now that $\mathfrak{a}$ is an irreducible left ideal, that is, that it contains no sub-ideal other than itself and the null ideal. Then none of the quantities $e_i u_1 e_j = e_i \bar{e}_j u_1$ are zero. For $e_i \bar{e}_i u_1$ is certainly not zero, and the set

$$e_1 \bar{e}_i u_1 \ldots\ldots e_f \bar{e}_i u_1$$

forms an ideal basis for an ideal obtained from $\mathfrak{a}$ by the operator endomorphism e_i, which must, on account of the irreducibility, be an isomorphism.

It does not follow without restrictions on K, that $\mathfrak{a}$ is necessarily one-sided, but in this case the above quantities express a basis for the two-sided ideal generated by $\mathfrak{a}$. Let this be denoted by $\mathfrak{v}$, and let $\mathfrak{w}$ denote the sub-ideal of $\mathfrak{b}$ orthogonal to $\mathfrak{v}$.

One can construct a matrix ring isomorphic with the remainder-class ring $\mathfrak{O}/\mathfrak{w}$ in the following manner.

For any substitution τ, $\tau^{-1} e_i$ is in $\mathfrak{a}$ and therefore uniquely expressible in terms of the basis elements.

$$\tau^{-1} e_i = \sum_j \lambda_{ij}^{\tau} e_j$$

Then, by the independence of the basis,

$$\tau^{-1} \sigma^{-1} e_i = \sum_j \sum_k \lambda_{ik}^{\sigma} \lambda_{kj}^{\tau} e_j = (\sigma\tau)^{-1} e_i$$

implies that

$$\sum_k \lambda_{ik}^{\sigma} \lambda_{kj}^{\tau} = \lambda_{ij}^{\sigma\tau}$$

so that the matrices $[\lambda_{ij}^{\tau}]$, $\tau = s_1, \cdots s_{n!}$; $i, j = 1 \cdots f$, represent the group.

Moreover, since the basis elements $\{e_1 \cdots e_f\}$ are orthogonal the same is true of $\tau^{-1} e_r = \sum_j \lambda_{rj}^{\tau} e_j$, $r = 1, 2 \cdots f$.

But, if K is in particular the field of real numbers, the basis elements $e_1 \ldots\ldots e_f$ can always be chosen so that, expressed in terms of the quantities $u_1 \ldots\ldots u_{n!}$, they form a normal set. This operation does not affect the substitutional properties. If the basis has been normalised in this way then the matrices $[\lambda_{ij}^{\tau}]$ are obviously all orthogonal. Written explicitly, this fact may be seen /

seen by writing $e_i = \sum u_{i\sigma} \cdot \sigma$ so that the last orthogonality relation becomes

$$\sum_{\sigma} \left\{ \left(\sum_{j} \lambda^{\tau}_{rj} u_{j\sigma} \right) \left(\sum_{j} \lambda^{\tau}_{r'j} u_{j\sigma} \right) \right\} = \delta_{rr'}$$

where $\delta_{rr'}$ is zero unless $r = r'$, in which case it is unity, and this may be written

$$\sum_{j} \sum_{\sigma} \lambda^{\tau}_{rj} \sum_{j'} u_{j\sigma} u_{j'\sigma} \cdot \lambda^{\tau}_{r'j'} = \delta_{rr'}$$

whence

$$\sum_{j} (\lambda^{\tau}_{rj})^2 = 1 \quad ; \quad \sum \lambda^{\tau}_{rj} \lambda^{\tau}_{r'j} = 0 , \quad r \neq r',$$

by the orthogonality of $\{ e_1 \cdots\cdots e_f \}$.

The correspondence $\tau \longrightarrow [\lambda^{\tau}_{ij}]$ therefore gives an orthogonal representation of the group which may be extended to a ring-homo-morphism in the manner described above. Orthogonality in the group representation corresponds to a relation between transposition and inversion in the ring.

$$a \longrightarrow A \quad ; \quad \bar{a} \longrightarrow A'$$

Also the relation $n\, e_i u_1 = 0$, $n \in \mathcal{N}$ proved above, combined with the minimal nature of the two-sided ideal $\mathcal{N}$, shows that the matric ring is isomorphic with $\mathcal{O}/_{\mathcal{N}}$

It is convenient, from the point of view of the matric reduction theory, to be able to obtain the representing matrices in another manner, or at least to show that the two representations considered below are identical. A little reflection on the manner in which a vector is expressed in terms of an orthogonal basis shows that the quantities λ^{τ}_{ij} can be expressed in terms of the $u_{i\sigma}$ by computing the scalar product of $\tau^{-1} e_i$ and the quantities e_j . The co-efficient of e_j in the expansion of $\tau^{-1} e_i$ is then

$$\sum u_{i,(\tau^{-1}\sigma)}\, u_{j\sigma}$$

which is precisely the co-efficient of τ in the product $e_i \bar{e}_j$. Hence

$$e_i \bar{e}_j = \sum_{\tau} \lambda^{\tau}_{ij} \cdot \tau$$

The matric consequences of this relation are discussed in paragraph 4.

It has been supposed that the left ideal $\mathfrak{a}$ is definitely a sub-ideal of $\mathfrak{r}$, and shown that in this case

$$\mathfrak{r} = \{ e_1\bar{e}_1, e_2\bar{e}_1 \cdots\cdots e_f\bar{e}_f \},$$

where the basis elements are ~~obviously~~ independent. It can be shown that they are actually orthogonal, since, on adding a basis for $\mathfrak{n}$ the whole ideal $\mathfrak{O}$ is covered and co-efficients K_{ij}^{τ} may be determined uniquely so that

$$\tau \equiv \sum_{i,j} K_{ij}^{\tau}\, e_i\bar{e}_j \quad (\mathfrak{n}).$$

Then the relations

$$e_r\bar{e}_s = \sum_{\tau, i, j} \lambda_{rs}^{\tau} K_{ij}^{\tau}\, e_i\bar{e}_j$$

show that

$$\sum_{\tau} \lambda_{rs}^{\tau} K_{ij}^{\tau} = \delta_{ir}\delta_{js},$$

while it is clear from the congruency relation that

$$\sum_{\tau} K_{ij}^{\tau} \cdot \tau \in \mathfrak{r}$$

It follows that $\sum_{\tau} K_{ij}^{\tau} \cdot \tau$ is at most a multiple of $e_i\bar{e}_j$ and then the above orthogonality relation shows the basis is orthogonal, though not of course normal. This result shows that the co-efficients λ_{ij}^{τ} form orthogonal sets in each of the sets i, j, τ taken separately, and that $\mathfrak{r}$ is covered by the f non-overlapping ideals

$$\mathfrak{l}_i = \{ e_1\bar{e}_i, e_2\bar{e}_i \cdots\cdots e_f\bar{e}_i \}, \qquad i = 1, 2 \cdots f,$$

as also by the non-overlapping right ideals

$$\mathfrak{r}_i = \{ e_i\bar{e}_1, e_i\bar{e}_2 \cdots\cdots e_i\bar{e}_f \}, \qquad i = 1, 2 \cdots f.$$

It is also an immediate consequence of this result, that, in the ring homo-morphism

$$e_i\bar{e}_j \longrightarrow \vartheta_{ij}\, e_{ij}, \qquad \vartheta_{ij} \in K,$$

where e_{ij}, $ij = 1, \cdots f$ denote the basis elements of the matrix ring. This relation gives an explicit form for the isomorphism between $\mathfrak{O}/\mathfrak{n}$ and the full matrix ring in K of matrices of order $f \times f$ and shows also that

$$e_i\bar{e}_j \cdot e_k\bar{e}_l = \vartheta \cdot \delta_{jk}\, e_i\bar{e}_l, \qquad \vartheta \in K,$$

the equality sign being valid since both sides belong to $\mathfrak{r}$.

The left-ideals considered above are, of course, not unique, whereas the two-sided ideals are. It is also clear from the /

the manner in which the matrices were obtained that a change in the basis $\{e_1, e_2 \cdots\cdots e_f\}$ within the same ideal yields an equivalent representation. Since an isomorphic left ideal admits a basis corresponding with the original in an operator-isomorphism, and since all left ideals of $\mathfrak{r}$ are isomorphic, the representation is altogether independent of the choice of $\mathfrak{a}$, and depends on the particular basis of $\mathfrak{a}$ only as far as equivalence is concerned. It is also obvious that a representation generated from an ideal in the manner described above is reducible if the ideal is reducible.

Conversely, given any matric representation of the group, and hence of the ring : $\tau \longrightarrow A_\tau = [\lambda^\tau_{ij}]$, the quantities

$$e_{\kappa j} = \sum \lambda^\tau_{\kappa j} \cdot \tau \quad , \quad \kappa = 1, 2 \cdots f ,$$

form an ideal, since

$$\sigma^{-1} e_{\kappa j} = \sum_\tau \lambda^{\sigma\tau}_{\kappa j} \cdot \tau$$

$$= \sum_r \lambda^\sigma_{\kappa r} \sum_\tau \lambda^\tau_{rj} \tau$$

and correspondingly the $n! - f^2$ linearly independent solutions of

$$\sum x_\tau A_\tau = 0$$

yield an orthogonal basis for all quantities represented by the null matrix in the homo-morphism. These therefore form an ideal in $\mathcal{O}$ — the analogue of the ideal $\mathcal{W}$ in the preceding section.

Left and right ideals can be obtained as row or column sets as described above and correspond to the reciprocal space to that of the $n! - f$ linearly independent solutions of vector equations corresponding to the matric one above. It is obvious that reducibility of the ideals would imply reducibility in the matrix representation.

A particular property of the quantities $\{e_1 \cdots e_f\}$ which will be useful in a later section arises when only one of basis elements e_j involves the ideality. It is clear that the basis can always be chosen in such a way that this will be true. If $b_\epsilon e_i = \delta_{if} u_{f1}$ then

$$\varepsilon = u_{f1} e_f + b = u_{f1} \cdot \bar{e}_f + \bar{b} \qquad , \quad b \in \mathfrak{b} .$$

Hence

$$e_i = u_{f1} e_i \bar{e}_f$$

Thus $u_{f1} \bar{e}_f$ is a right-unit of the left ideal, and $\{e_1 \cdots\cdots e_f\}$, apart from a constant factor, express the elements in the last column of the group matrix generated from the ideal basis $\{e_1 \cdots\cdots e_f\}$. The bases selected for the constructions given below will be chosen in such a way as to exhibit this property.

3. AN ILLUSTRATION OF THE PRECEDING RESULTS: $\mathfrak{h}_{[3]}$ AND THE RING $\mathcal{O}_3$.

As the development in later sections will proceed by induction it will be necessary to derive certain results concerning the groups of lowest order, and, the cases n = 1, 2 being trivial, the discussion of case n = 3 is carried out fully in this section; illustrating many of the results of the preceding theory.

Consider

$$\mathfrak{h}_{[3]} : \quad (13) \quad , (123) \quad , (23) \quad . (132) \qquad (12) \qquad 1$$

Corresponding to the ideals $S_{[3]}$, $S_{[3]}*$ there exist the representations of unit order:

$$\begin{array}{lcccccc} [3] : & 1 & 1 & 1 & 1 & 1 & 1 \\ [1^3] : & -1 & 1 & -1 & 1 & -1 & 1 \end{array}$$

where the permutations are taken in the order given above.

To obtain another irreducible ideal, consider the left ideal generated by $S_{[2,1]}$:

$$\ell = \left\{ (13) + (123) \ ; \ (23) + (132) \ ; \ (12) + 1 \right\}$$

The sum of the basis elements shows that this has $S_{[3]}$ as a sub-ideal. Clearly any quantity of ℓ is orthogonal to $S^*_{[1^3]}$ and, there being no other ideals of unit rank, the set orthogonal to $S_{[3]}$ must be irreducible. Hence

$$\mathfrak{a}_{[2,1]} = \{ e'_1 \ , e'_2 \}$$

where

$$e'_1 = -\{(13) + (123)\} + \{(23) + (132)\}$$

$$e'_2 = -\{(13) + (123)\} - \{(23) + (132)\} + 2\cdot\{(12) + 1\}$$

is an irreducible left ideal. To construct a representation from this set normalise the basis with respect to the group elements, considering instead

$$\mathfrak{a}_{[2,1]} = \{ e_1 \ , e_2 \}$$

with

$$e_1 = \tfrac{1}{2}\, e'_1 \quad ; \quad e_2 = \frac{1}{2\sqrt{3}}\, e'_2$$

It will be seen later that, in the matric reduction of representa-:tions, this normalisation is a relative process, and it will, in fact, be necessary to use the same sets with different 'normalising' factors.

Representing/

Representing the above co-sets in order by a, b, c, the co-set multiplication table for products with their inverses:

	$\bar{a}$	$\bar{b}$	$\bar{c}$
a	$1 + (23)$	$(132) + (12)$	$(123) + (13)$
b	$(123) + (12)$	$1 + (13)$	$(132) + (23)$
c	$(132) + (13)$	$(123) + (23)$	$1 + (12)$

enables one to read off immediately

$$e_1\bar{e}_1 = \tfrac{1}{2}\{(13) - (123) + (23) - (132) - 2\cdot(12) + 2\cdot 1\}$$
$$e_1\bar{e}_2 = \tfrac{\sqrt{3}}{2}\{-(13) - (123) + (23) + (132)\}$$
$$e_2\bar{e}_1 = \tfrac{\sqrt{3}}{2}\{-(13) + (123) + (23) - (132)\}$$
$$e_2\bar{e}_2 = \tfrac{1}{2}\{-(13) - (123) - (23) - (132) + 2\cdot(12) + 2\cdot 1\}$$

and to write down the individual matrices of the representation as

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 & \cdot \\ \cdot & 1 \end{bmatrix}, \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix},$$

taken in the same order as the substitution above. It is to be observed that the sub-group $\mathfrak{h}_{[2]} = 1 + (12)$ of substitution not involving the symbol 3 is represented by the direct sum of two scalar representations of that group. Also as shown in the general theory,

$$e_1 \cdot \bar{e}_2 = \sqrt{3} \cdot e_1$$
$$e_2 \cdot \bar{e}_2 = \sqrt{3} \cdot e_2$$

where $u_{[2,1]} = \frac{1}{\sqrt{3}}$

Since $(e_1\bar{e}_1)^* = e_2\bar{e}_2$; $(e_1\bar{e}_2)^* = -e_2\bar{e}_1$, the two left ideals transform into themselves under the conjugate isomorphism. Correspondingly the representation obtained from that given above, by changing the signs of all matrix elements in those matrices representing odd substitutions, can also be obtained from it by transforming with the matrix

$$\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$$

The matrices have been obtained above by means of the second method outlined in the general theory, but it is easily verified that they could be generated from the ideal $\mathfrak{A}_{[2,1]}$ in the manner already described.

Also/

Also

$$e_1 \rightarrow \begin{bmatrix} \cdot & \sqrt{3} \\ \cdot & \cdot \end{bmatrix}, \qquad e_2 \rightarrow \begin{bmatrix} \cdot & \cdot \\ \cdot & \sqrt{3} \end{bmatrix}$$

THE SYMBOLISATION AND REDUCTION OF MATRIC REPRESENTATIONS.

It has been shown that to every irreducible matric representation of $\mathfrak{H}_{[n]}$ there corresponds a two-sided ideal in $\mathcal{O}_n$ and that this ideal is minimal and expresses reciprocally the original representation. It follows that, apart from equivalent sub-matrices in direct sum and from equivalent forms, every matrix representing the group can be obtained by suitable transformation of that corresponding to the ideal $\mathcal{O}$ itself, i.e. of the Frobenius regular representation. To obtain this the orthogonal normal basis $\mathcal{O} = \{ s_1, s_2 \ldots\ldots s_{n!} \}$ is taken and the matrices are then to be defined by

$$\tau^{-1} s_i = \sum \lambda^{\tau}_{ij} \, s_j$$

whence

$$\lambda^{\tau}_{ij} = \delta_{jk}, \quad \tau = s_i s_k^{-1}$$

The theory of the reduction of this matrix is actually embodied in the results of paragraph 2, and the present section is concerned with building up a simple isomorphism between the ideal theory and the actual reducing matrices.

To perform a simultaneous reduction of the n! matrices it is convenient to consider instead the group matrix obtained by multiplying the matrix representing s_i by u_i and summing over all i. The quantities $u_1 \ldots\ldots u_{n!}$ being independent, the two problems are equivalent. Then, as in paragraph 1, one may write

$$u_k = a_{j_1 i_1} a_{j_2 i_2} \ldots\ldots a_{j_n i_n} \qquad s_i s_j^{-1} = s_k$$

and extend this notation further by separating the sets of suffixes.

$$u_k = a_{1 i_1} \ldots\ldots a_{n i_n} \alpha_{1 j_1} \ldots\ldots \alpha_{n j_n} = a_i \alpha_j$$

where the last form given will be a convenient abbreviation. The group matrix then becomes $[a_i \alpha_j]$.

Now it is not only true that the symbols a_i are the same for all elements in the same row in this matrix, for this same statement is valid after the matrix has been pre-multiplied by any other. In fact, if sums over the separate symbols a_i and α_j are defined in the obvious manner, to satisfy the distributive law, the pre-multiplication by any matrix $[\mu_{ij}]$ puts in place $\{i, j\}$ the quantity $\left(\sum \mu_{ir} a_r \right) \cdot \alpha_j$.

Moreover, the column symbols α_j are unaffected in this operation and correspondingly in post-multiplication the combinations of remain unaltered. In fact post-multiplication by the transposed matrix gives in place $\{i, j\}$

$$\left(\sum_r \mu_{ir} a_r\right)\left(\sum_s \mu_{js} \alpha_s\right)$$

These remarks merely express the fact that matrix pre-multiplication can form only linear combinations of rows, and similarly post-multiplication only combination of columns. They serve, however, to give a clear explanation of the advantages of the orthogonal reduction, in which the combinations of row and column symbols correspond. Expressed in terms of the ring as multiplication domain the above expression for the quantity in place $\{i, j\}$ becomes

$$e_i u_1 e_j = e_i \bar{e}_j u_1$$

where $$e_i = \sum_r \mu_{ir} s_r$$

It has been shown that, if $\{e_1 \ldots e_f\}$ is a basis for a left ideal with a reciprocal ideal $\mathfrak{b}$ then

$$e_i \bar{b} = b\bar{e}_i = 0 \qquad b \in \mathfrak{b}$$

If then e_i has the form given above and the elements of the matrix $[\mu_{ir}]$ $i = 1, 2 \ldots f$; $r = 1, 2 \ldots n!$ are chosen for the first f rows of a pre-multiplying matrix, $e_1 \ldots e_f$ being orthogonal and normal, and this completed to an $n! \times n!$ matrix by adding the co-efficients of any basis for $\mathfrak{b}$ the result of transforming the group matrix of the regular representation with this will be a matrix in which the group matrix of the representation corresponding to the ideal $\{e_1 \ldots e_f\}$ is isolated in the first f rows and columns. Since also the sets

$$\{e_1\bar{e}_j, e_2\bar{e}_j \ldots e_f\bar{e}_j\}, \quad j = 1, 2 \ldots f,$$

are mutually orthogonal ideals the above considerations will apply to a pre-multiplier of which the first f^2 rows contain the f sets of $f \times n!$ normalised co-efficients of the quantities $e_i\bar{e}_j$ and, by virtue of the relation obtained in paragraph 2 for these sets, the resulting matrix contains the direct sum of the group matrix corresponding to $\{e_1 \ldots e_f\}$ repeated f times, together with an additional matrix corresponding to $\mathfrak{w}$.

If also $\mathfrak{w}$ is expressed in a basis containing sub-sets for mutually orthogonal left ideals, then the complete reduction of the group matrix to a direct sum of irreducible group matrices can be effected by this method.

Thus in case n = 3, taking as before the order

(13) (123) (23) (132) (12) 1

the /

the orthogonal matrix

$$H_{[1^3]} = \left[\begin{array}{cccccc} \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} \\ \hdashline \frac{1}{2\sqrt3} & -\frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & -\frac{1}{2\sqrt3} & -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & & \\ \hdashline -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & & \\ -\frac{1}{2\sqrt3} & -\frac{1}{2\sqrt3} & -\frac{1}{2\sqrt3} & -\frac{1}{2\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ \hdashline -\frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & \frac{1}{\sqrt6} \end{array}\right] \begin{array}{c} {[3]} \\ {[21]} \\ \\ {[21]} \\ \\ {[1^3]} \end{array}$$

reduces the regular representation completely to the direct sum

$$A_{[3]} \dotplus A_{[2,1]} \dotplus A_{[2,1]} \dotplus A_{[1^3]} .$$

THE DIRECT CONSTRUCTION OF ORTHOGONAL REPRESENTATIONS.

In this section a unique form for the basis $\{e_1 \cdots e_f\}$ corresponding to each irreducible two-sided ideal in $\mathcal{O}$, will be obtained. The basis will have the form discussed at the end of paragraph 2, so that, before normalisation, it will exhibit the elements of the last column of the orthogonal group matrix, the complete form of this being then obtained as products $e_i \bar{e}_j$.

The discussion will be restricted to the field of real numbers, although a corresponding, but less symmetric, rational form will be implied. If, then, the reduction can be performed completely in this field, the total number of representations obtained will be equal to the rank of the centrum, i.e. to the number of partitions of n. It is natural therefore to consider the two-sided ideals generated from the various quantities $S_{[\lambda]}$ Arranging these in ascending order of their ranks, the complete reduction in any field will follow if it is possible to construct a sub-ideal $\mathfrak{a}_{[\lambda]}$, for each value of $[\lambda]$, which annihilates all quantities of the two-sided ideals of lower order, or even if it can be shown that there exist quantities of $(S_{[\lambda]})$ not expressible in terms of the quantities of earlier ideals. In this latter form the result is obvious from symmetry considerations, whereas in the former it is one of the preliminary results of Young's investiga-:tions, since his operators N P satisfy the required condition. This might also be regarded as a consequence of the inter-expressibility of symmetric functions as used in investigating trace relations, since N P belongs to both the ideal of $S_{[\lambda]}$ and to the inverse conjugate ideal, and hence to their intersection, the irreducible $\mathfrak{a}_{[\lambda]}$.

It remains, then, to construct on each ideal $(S_{[\lambda]})$ a basis annihilated by or, equivalently, orthogonal to all $(S_{[\mu]})$ with $[\mu] < [\lambda]$ The discussion proceeds by the method of induction. Let it be supposed therefore that such a construction has been performed for all partitions of $n-1, n-2, \cdots\cdots$ and, in particular, that these earlier /

earlier representations all exhibit the form involving the last column of their group matrices.

To obtain a basis for the induction it is necessary to consider the behaviour of an ideal in $\mathcal{O}_n$ under the restricted operator-domain $\mathcal{O}_{n-1}$, and to develop an operational correspondence between $\mathcal{O}_{n-1}\mathfrak{M}_n$ and $\mathcal{O}_{n-1}\mathfrak{M}_{n-1}$

It will be convenient to develop first a notation adapted to the discussion of co-sets of $\mathfrak{H}_{[\lambda]}$, and derived by dividing the numbers 1, 2 .... n into sets of $\lambda_1, \lambda_2 \cdots\cdots \lambda_h$ elements. Accordingly we write first $\lambda_{ij} = \lambda_1+\lambda_2+\cdots+\lambda_{i-1}+j;\ i=1\cdots h;\ j=1\cdots\lambda_i$ Then, introducing new sets of symbols $\mu_{ij}, \nu_{ij}\cdots;\ i=1\cdots h;\ j=1\cdots\lambda_i$ conforming with the λ_{ij}, it will be convenient to write

$$\tau_\mu = s_i = \binom{1\ 2 \cdots n}{i_1\, i_2 \cdots i_n}$$

if $i_{\lambda_{ij}} = \mu_{ij}$.

In this new notation every substitution is uniquely expressed as a τ_μ for some μ, and conversely. In particular, τ_λ represents the identity. Moreover, the $[\lambda]!$ elements of the co-set of $\mathfrak{H}_{[\lambda]}$ which contains τ_μ are represented by the different re-arrangements between numbers $\mu_{ij}, \mu_{ij'}$. Forming the sum over these we obtain quantities independent of the order of the second suffices

$$\eta_\mu = \tau_\mu \cdot S_{[\lambda]}$$

in which only $\binom{n}{\lambda}$ different values of μ are required to form a basis for the ideal $(S_{[\lambda]})$.

Let $[\lambda^i]$ denote the partition $[\lambda_1, \lambda_2 \cdots \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1} \cdots \lambda_h]$ of n-1 and let η_{μ^i} be a typical co-set summation in which n appears among the symbols μ_{ij}. The full basis for the ideal $(S_{[\lambda]})$ in terms of distinct η_μ then breaks up into h parts corresponding to quantities of type $\eta_{\mu^1} \cdots\cdots \eta_{\mu^h}$.

It will be convenient to denote by $M_1, M_2 \cdots\cdots M_h$ the sets in $(S_{[\lambda]})$ consisting of all linear combinations of corresponding η_{μ^i}. Then any quantity of the left ideal falls uniquely into a sum of quantities from each of the sets M_i and M_i has a basis of order $\binom{n}{\lambda^i}$.

Now, on applying to the basis elements η_μ of $(S_{[\lambda]})$ the restricted group of operators $\mathfrak{H}_{[n-1]}$ it is at once obvious that the sets M_i are not merely invariant sub-moduls, but are actually isomorphic with the ideals $(S_{[\lambda^i]})$ of $\mathcal{O}_{n-1}$. In fact the representative of η_{μ^i} can be chosen with $i_{\lambda_{i\lambda_i}} = n$ and the operator isomorphism for M_i reduces to the endomorphism

$$\begin{pmatrix} n & \lambda_{i,\lambda_i} & \cdots\cdots & n-1 \\ \lambda_{i,\lambda_i} & \lambda_{i+1,1} & \cdots\cdots & n \end{pmatrix}$$

of the group ring. We shall represent the isomorphism by writing

$$\eta_{\mu^i} \longrightarrow \eta_{\bar{\mu}^i}.$$

The left ideal $\mathfrak{A}_{[\lambda]}$ which is now to be constructed is defined by its orthogonality to all the two-sided ideals of earlier partitions. If the sum of the intersections of these with the left ideal $(S_{[\lambda]})$ be /

be denoted by $\mathcal{N}_{[\lambda]}$, then $\mathcal{O}_{[\lambda]}$ may be defined as the set of all quantities of $(S_{[\lambda]})$ orthogonal to $\mathcal{N}_{[\lambda]}$. Accordingly we investigate first the form of $\mathcal{N}_{[\lambda]}$.

A preliminary reduction can be obtained by remarking that, since the sums and intersections of ideals are again ideals, it is sufficient to consider each value of $[\mu]$ separately. Then, since the intersection of any left ideal in the two-sided $(S_{[\mu]})$ with the full right ideal generated by $S_{[\mu]}$ is non zero, a generation of the form $S_{[\mu]}\cdot e$ can be obtained for the left ideal common to the left ideal $(S_{[\lambda]})$ and the two-sided $(S_{[\mu]})$. Moreover one can ensure, by left multiplication, that the identity is involved and in fact that it has co-efficient unity. The common ideal is therefore to be obtained by forming left multiples of a quantity which is at once a combination of left co-sets of $\mathfrak{h}_{[\lambda]}$ and right co-sets of $\mathfrak{h}_{[\mu]}$, and one which involves the identity and therefore all elements of $\mathfrak{h}_{[\lambda]}$ and $\mathfrak{h}_{[\mu]}$ without a multiplier in K. One should then proceed to seek the elements of $\mathfrak{h}_{[\lambda]}$ among the right co-sets of $\mathfrak{h}_{[\mu]}$ and similarly those $\mathfrak{h}_{[\mu]}$ among the left of $\mathfrak{h}_{[\lambda]}$, giving all elements of the co-sets obtained the co-efficient 1. Completing this process in the particular case where $[\mu] = [\lambda_1 \cdots \lambda_{i-1}, \lambda_i+1, \cdots \lambda_k-1 \cdots \lambda_h]$ and where an obvious transformation has been applied to the symbols in $\mathfrak{h}_{[\mu]}$ one obtains an expression of the form

$$\{\varepsilon + \sum_j (\lambda_{ij}\lambda_{k1})\}\, \eta_\lambda$$

obviously common to the two ideals, and covering, in its left multiples, the expressions which would be obtained from those facts in the preceding argument which do not necessarily carry a unit co-efficient. Obviously a $[\mu]$ which differs from $[\lambda]$ by more than an interchange of the above type can yield only common parts consisting of combinations of those of the type given above, and hence this sum, extended, for each $i<k$, to a left ideal basis, yields a complete basis for $\mathcal{N}_{[\lambda]}$. Although this will, in general, include redundant elements it is convenient to use the simpler form.

When considered in relation to the sets $M_1, M_2 \cdots M_h$ it appears that the basis elements of $\mathcal{N}_{[\lambda]}$ assume two distinct forms. Since

$$\begin{aligned}\tau_{\mu\rho}\{\varepsilon + \sum_j (\lambda_{ij}\lambda_{k1})\}\eta_\lambda &= \{\varepsilon + \sum_j (\mu_{ij}\mu_{k1})\}\, \tau_{\mu\rho}\, \eta_\lambda \\ &= \{\varepsilon + \sum_j (\mu_{ij}\mu_{k1})\}\, \eta_{\mu\rho}\end{aligned}$$

it is clear that this quantity belongs entirely to M_ρ, unless either $i = \rho$ or $k = \rho$ and $\mu_{k1} = n$. Hence the basis of $\mathcal{N}_{[\lambda]}$ imposes two types of conditions on the elements of $\mathcal{O}_{[\lambda]}$. In the first type sums over co-efficients in each M_ρ taken separately are to vanish, while in the second the co-efficient of each $\eta_{\mu\rho}$ in M_ρ is expressed as a sum taken over the co-efficients of the parts in an earlier M_σ. Since such a sum exists for each $\sigma<\rho$ it follows in particular that if the part of a quantity of $\mathcal{O}_{[\lambda]}$ contained in a certain M_ρ is not zero, that belonging to an earlier M_σ cannot be zero.

The above analysis shows also that the conditions of the first /

first type which apply to M_h go over, in the isomorphism with $\mathcal{O}_{n-1}$ to the totality of conditions applying to $\mathfrak{A}_{[\lambda^h]} \subset \mathcal{O}_{n-1}$. It follows that, in the basis of $\mathfrak{A}_{[\lambda]}$, the parts belonging to M_h correspond completely to a basis for $\mathfrak{A}_{[\lambda^h]}$. The $\mathfrak{A}_{[\lambda]}$ basis can therefore chosen so that f^h go over to an orthogonal basis for $\mathfrak{A}_{[\lambda^h]}$, and the remaining $f-f^h$ do not involve quantities of M_h. It will be convenient to denote by E_h a typical quantity of the former type, and by e_h^i the intersection of E_h and M_i.

If it were possible to reverse the conditions of the second type one could now derive the e_h^i from e_h^h and so obtain E_h. This can only be done in the most trivial cases (e.g. $[\lambda] = [1^2]$, $[2^2]$ ...) since in general other basis elements of $\mathfrak{A}_{[\lambda]}$ are involved. But, since it is in general possible to reduce the set of matrices corresponding to the substitutions of $\mathfrak{H}_{[n-1]}$ in the representation of type $[\lambda]$, it is correspondingly possible to express the ideal obtained directly from the matric representation, and so the isomorphic ideal $\mathfrak{A}_{[\lambda]}$, as a direct sum of ideals relative to the restricted operator-domain. To construct a basis of this type one would require that e_h^i and e_h^h go over, in the $\mathfrak{H}_{[n-1]}$ isomorphism, into isomorphic ideals, and therefore that $\tilde{e}_h^i$ should be an ideal contained in $(S_{[\lambda']})$ and isomorphic with $\mathfrak{A}_{[\lambda^h]}$. What is then required to complete the construction is a second operator isomorphism to relate quantities of M_i with those of later M_j.

It will also be required that orthogonality be preserved under the correspondence and that the sets e_h thus determined will satisfy all the conditions imposed for orthogonality with quantities of $\mathcal{N}_{[\lambda]}$.

Before developing such an isomorphism consider first the situation which now arises in the terms of M_{h-1} corresponding to basis elements not included in the E_h.

Firstly if $\lambda_{h-1} = \lambda_h$ then the sum

$$\left\{ \varepsilon + \sum_j (\mu_{hj}\, n) \right\} T_{\rho^{h-1}}$$

relates $[\lambda]$ to an earlier partition, so that the conditions can be reversed and the part in M_{h-1} written down directly from that in M_h. But this latter being zero, the part in M_{h-1} is also zero and hence no new sets E_{h-1} are obtained.

Now let λ_ρ, possibly with $\rho = h-1$, be the smallest partition part with $\lambda_\rho \neq \lambda_h$ and consider

$$\left\{ \varepsilon + \sum_j (\mu_{ij}\, \mu_{kl}) \right\} T_{\rho^\rho} \qquad i < k.$$

The conditions of the first kind go over in the isomorphism to conditions of $\mathcal{N}_{[\lambda']}$. But for those quantities of $\mathfrak{A}_{[\lambda]}$ with no parts in $M_{\rho+1} \dots M_h$ those of the second kind corresponding to $i = \rho$ now augment the first, since the addition of terms from $M_{\rho+1} \dots M_h$ merely adds zero, and the conditions correspond to a complete /

complete basis for $\mathfrak{M}_{[\lambda^\rho]}$. By the same argument as was used for E_h we therefore obtain f^ρ new quantities for the basis, where f^ρ is the order of $\mathfrak{A}_{[\lambda^\rho]}$. Let a typical one be denoted by

$$E_\rho = e_\rho^1 + \ldots\ldots + e_\rho^\rho ,$$

with $e_\rho^i \subset M_i$, in which the f^ρ quantities e_ρ^i form an orthogonal basis for quantities in M_ρ corresponding in the isomorphism with $\mathfrak{A}_{[\lambda^\rho]}$

Proceeding in this manner it appears that $\mathfrak{A}_{[\lambda]}$ will have a basis of

$$f^1 + f^2 + \ldots + f^\rho + \ldots + f^h \qquad \lambda_\rho \neq \lambda_{\rho+1}$$

quantities of types represented by $E_1 \ldots E_\rho \ldots E_h$ with

$$E_\rho = e_\rho^1 + \ldots\ldots + e_\rho^\rho \qquad\qquad e_\rho^i \; \varepsilon \; M_i$$

and with the e_i^j orthogonal to each other, for each value of i.

It remains now to show that the non zero quantities

$$e_\rho^i \qquad\qquad i = 1, 2 \ldots\ldots \rho-1$$

can be constructed for each ρ in such a way as to satisfy all conditions arising from $\mathfrak{M}_{[\lambda]}$ and to exhibit orthogonality in the sets $\{e_i^1 \ldots\ldots e_\rho^i \ldots\ldots e_h^i\}$ and isomorphism with respect to the operator-domain $\mathfrak{h}_{[n-1]}$ in the sets $\{e_\rho^1 \ldots\ldots e_\rho^\rho\}$.

Consider the correspondence

$$\eta_{\mu\rho} \longrightarrow c_i^r \sum_{\kappa=1}^{\lambda_i} (n\mu_{i\kappa})\, \eta_{\mu\rho} \qquad\qquad \rho > i$$

by means of which any quantity of $M_{i+1} + \ldots\ldots + M_h$ expresses a definite quantity of M_i. The relation is an operational one for $\mathfrak{h}_{[n-1]}$ since, if $\tau = \begin{pmatrix} \mu_{ij}^\rho \\ \nu_{ij}^\rho \end{pmatrix} \varepsilon \; \mathfrak{h}_{[n-1]}$

$$\text{then} \qquad \tau . \eta_{\mu\rho} = \eta_{\nu\rho} \longrightarrow c_i^r \sum_\kappa (n\nu_{i\kappa})\, \eta_{\nu\rho}$$

$$\# \qquad = c_i^r \, \tau . \sum_\kappa (n\mu_{i\kappa})\, \eta_{\mu\rho} .$$

c_i^r is a constant for each set E_r and each M_i and will be determined later.

The remaining part of the discussion conveniently falls into four separate sections.

A: If the quantities $e_r^j, e_s^j \ldots , j = i+1, \ldots r$ are orthogonal, though not necessarily normalised, in the sets with j fixed, if they have been derived successively by means of the above correspondence, and if they satisfy the conditions of the first kind belonging to their sets, and those of the second kind relating them to later sets, then, if, in the correspondence,

$$e_r^{i+1} + e_r^{i+2} + \ldots\ldots + e_r^r \rightarrow e_r^i$$

and if $e_s^i \ldots$ are derived similarly, the $f^r + f^s + \ldots$ quantities $e_r^i, e_s^i, \ldots\ldots$ are non-zero and orthogonal.

For consider any two quantities of M_i

$$\sum \kappa_{\mu^i}\, \eta_{\mu^i}, \quad \sum \kappa'_{\mu^i}\, \eta_{\mu^i}$$

of types r, s respectively, which have been derived in the manner described, and let the corresponding sets from which they have been derived be denoted by

$$\sum_{\rho,\nu} \kappa_{\nu\rho}\, \eta_{\nu\rho}, \quad \sum_{\rho,\nu} \kappa'_{\nu\rho}\, \eta_{\nu\rho}.$$

Then the co-efficient of $\kappa'_{\nu\rho}$ in $\sum_{\mu} \kappa_{\mu^i} \kappa'_{\mu^i}$ is the sum of $c_s^i\, \kappa_{\mu^i}$ taken over the λ_i values of μ for which η_{μ^i} assumes the form

$$(n\, \nu_{i\kappa}^{\rho})\, \eta_{\nu\rho} \qquad \kappa = 1, 2 \cdots \lambda_i.$$

Expanding these κ_μ^i in terms of co-efficients from later M_j we distinguish three parts requiring different treatment.

(i): $i < j < \rho$:

We require here $c_r^i \cdot c_s^i$ multiplied by the sum of the co-efficients corresponding to the co-sets

$$(n\, \nu_{jl}^{\rho})(n\, \nu_{i\kappa}^{\rho})\, \eta_{\nu\rho} \qquad \kappa = 1 \cdots \lambda_i;\ l = 1 \cdots \lambda_j; \quad j = i+1, \cdots \rho - 1$$

Writing the co-set as

$$(\nu_{jl}^{\rho}\, \nu_{i\kappa}^{\rho})(n\, \nu_{jl}^{\rho})\, \eta_{\nu\rho}$$

and summing first over κ, the result is minus the sum over the co-efficients of

$$(n\, \nu_{jl}^{\rho})\, \eta_{\nu\rho} \qquad l = 1, 2 \cdots \lambda_j; \quad j = i+1 \cdots \rho - 1$$

by a condition of the first kind in M_j. Summing now over l and repeating the result for different values of j, a condition of the second kind gives finally $c_r^i\, c_s^i\, (\rho - i - 1)\, \kappa_{\nu\rho}$.

(ii): $\rho < j \leq r$:

The summation over κ can be performed exactly as before and then the second, taken over j and l together, follows exactly the manner in which $\kappa_{\nu\rho}$ is defined, so that the contribution of this part is

$$-\frac{c_r^i\, c_s^i}{c_r^{\rho}}\, \kappa_{\nu\rho}$$

(iii): $j = \rho$:

Here one inversion for each of the λ_i values of μ leads back to $\kappa_{\nu\rho}$ directly, while each other inversion, summed over the different /

different values of ν yields $-K_{\nu\rho}$ by a condition of the first kind on M_ρ . Hence the total contribution is

$$c_r^i \, c_s^i \, (\lambda_i - \lambda_\rho + 1) \, K_{\nu\rho} \, .$$

Summing over these parts we obtain

$$c_r^i \, c_s^i \left\{ -\frac{1}{c_r^i} + \lambda_i - \lambda_\rho + \rho - i \right\} K_{\nu\rho}$$

and a final summation over $\nu\rho$, after multiplying by $K'_{\nu\rho}$ shows that the set is orthogonal. The argument does not involve any restriction on r and s and therefore yields the full result.

If, in the preceding, $r = s$, then the calculation gives the factor required for the normalisation of e_r^i . The actual choice of c_r^i will be made later, but it might be observed now that, as these numbers will all be negative, the above quantity is positive, and hence none of the e_r^i vanish.

B: Under the suppositions enumerated under A, the quantities e_r^i .... satisfy all conditions of the first kind and all of the second relating them to later M_ρ . They are also orthogonal to the quantities e_i^i , if these exist.

For the conditions arising from $\mathcal{W}_{[\lambda]}$ it must be shown that the sum over the co-efficients K_μ and those corresponding to

$$(\mu_{jk}\, \mu_{\rho\sigma})\, \eta_\mu \qquad\qquad K = 1 \ldots\ldots \lambda_j \; ; \; j < \rho, \text{ fixed},$$

is zero.

The discussion again breaks up into a number of different cases which may be enumerated thus:
(i): $j < i$ — of the first kind.

Since it has been shown that the e_r^i form a non-trivial set of orthogonal vectors isomorphic with e_r^r with respect to $\mathfrak{h}_{(n-1)}$ and since the conditions of the type considered correspond to quantities which the ideal $(\mathcal{S}_{[\lambda]})$ has in common with an earlier such ideal, it follows that, in the isomorphism $\eta_{\rho^i} \to \eta_{\bar\rho^i}$, the quantities e_r^i go over into quantities of the two-sided ideal of type $[\lambda^r]$ whereas the sums considered go over into a part of the earlier two-sided ideal in which λ_j is increased by one. This implies the orthogonality.

The same argument, applied instead to the irreducible ideals of type $[\lambda^i]$ to which the quantities e_i^i correspond in the isomorphism, shows them orthogonal to the e_r^i .

(ii): $j > i$ — (of the first kind)

Let the co-efficients involved be expressed in terms of those belonging to later M_ℓ and consider first the particular inversion $(n\mu_{\rho'\sigma'})$ and the co-efficients arising from its application to the various /

various τ_μ when $\rho' \neq j, \rho$. Each such sum expresses a condition of the first kind in $M_{\rho'}$ and is therefore zero. The same is also true in case $\rho' = \rho$ and $\sigma \neq \sigma'$

There remain $\lambda_i + 1$ inversions for each of the $\lambda_i + 1$ sets. Each inversion summed separately over the sets corresponds to a condition of the second kind relating to M_j and M_ρ and hence this condition is also satisfied.

<u>(iii)</u>: $j = i$ — conditions of the second kind in later M_ρ :

The discussion in this case follows the lines of the argument used in proving the orthogonality. Fix the attention first on the co-efficients of e_r^i involved and consider a fixed inversion $(n\, \mu_{\rho'\sigma'})$ arising in their extended form, supposing first that $\rho \neq \rho'$. The sum involved yields $-c_r^i$ multiplied by the co-efficient corresponding to the inversion $(n\mu_{\rho'\sigma'})$ applied to the one co-set μ^ρ, in M_ρ, under consideration, by a condition of the first kind.

If, then, $\rho' < \rho$ a condition of the second kind applied to the sum over σ' refers the discussion again to τ_{μ^ρ} and the sum over all such ρ' yields $c_r^i (\rho - i - 1)\, K_{\mu^\rho}$,

K_μ^ρ being the co-efficient of τ_{μ^ρ} in e_r^ρ.

On the other hand if $\rho' > \rho$ the sum over all the inversions

$$(n\mu_{\rho'\sigma'})\ \tau_{\mu^\rho}$$

taken both over ρ' and σ' relates back to the definition of K_{μ^ρ} and contributes the quantity

$$-\frac{c_r^i}{c_r^\rho} K_{\mu^\rho} \qquad \rho \neq r$$

Finally, when $\rho' = \rho$ we return to the original form and obtain inversion leading directly to K_{μ^ρ}. The others, corres-:ponding to $\sigma' \neq \sigma$ and summed over μ with σ' fixed, each relate back to K_{μ^ρ} on applying a condition of the first kind in M_ρ Hence the total contributions from $\rho' = \rho$ is

$$c_r^i (\lambda_i - \lambda_h + 1)$$

It should perhaps be added to this argument that the case $\rho > r$ does not require any special treatment, since conditions of the second kind apply to the quantities e_r^r.

Summing the various parts one finds that the construction does actually give an orthogonal basis for $\mathfrak{A}_{[\lambda]}$ if the e_r^i can be chosen so that

$$\frac{1}{c_r^i} = \frac{1}{c_r^\rho} - (\lambda_i - \lambda_\rho + \rho - i) \quad , \quad \rho = i+1, \ldots\ldots r-1$$

and /

and this will be valid for the various values of ρ if

$$C_r^i = -\frac{1}{\lambda_i - \lambda_r + r - i}$$

this being the form required in the case $\rho = r$ since the term involving $\frac{1}{C_r^\rho}$ then falls away.

It follows from the above discussion that the set of $f^1 + \cdots + f^h = f$ quantities $E_1 \cdots E_h$ provide an orthogonal basis for $\mathcal{O}_{[\lambda]}$. Whether the E_r appear in normalised form or not will depend upon the choice of the e_r^r. In particular, if these are all separately normalised reference to the results of A shows that the normalising factor required to reduce E_r to normal form is

$$\prod_{i=1}^{r-1} \sqrt{\frac{\lambda_i - \lambda_r + r - i}{\lambda_i - \lambda_r + r - i + 1}}$$

If the basis $E_1 \cdots E_h$ is to be used to generate the matric representation by use of the operator domain $\mathfrak{h}_{[n]}$ or equivalently by constructing from it a matrix with which to reduce the permutational representation of the ideal $(S_{[\lambda]})$ then this factor must be inserted. On the other hand, if $E_1 \cdots E_h$ are to be regarded as elements of the last column of a group matrix - a situation which arises when e_h^h exhibits this form - then the above type of normalisation no longer applies and one requires that, in particular, $K_\lambda = 1$.

THE CANONICAL FORM AND ITS $(n.\ n-1)$ INVERSIONS.

It will be supposed now that the quantities e_r^r appearing in $E_1 \cdots E_h$ correspond in the isomorphism with $\mathcal{O}_{n-1}$ to quantities derived in the same manner as $E_1 \cdots E_h$ and that this is continued for successive reductions in the order of the group ring. If in addition the condition of normality be imposed, the sets are uniquely determined.

In making the above suppositions on e_r^r one can distinguish between quantities E_r derived from e_r^r corresponding to different sub-ideals with respect to the restricted operator domain $\mathfrak{h}_{[n-2]}$ and write these $E_{r1} \cdots E_{rk_r}$, E_{rs} appearing among these if $\lambda_s^r \neq \lambda_{s+1}^r$. Correspondingly the sets M_i can be sub-divided in the second isomorphism into sets $M_{i1} \cdots M_{ih}$. Then, with an obvious extension of the notation, each E_{rs} can be written

$$E_{rs} = e_{rs}^{11} + e_{rs}^{12} + \cdots\cdots + e_{rs}^{rs}, \qquad e_{rs}^{ij} \in M_{ij},$$

there being altogether f^{rs} such expressions. If the notation be continued for successive reductions, the sets $rst\cdots$ cover the f lattice permutations /

permutations of type $[\lambda]$ written in reverse order, while the $i, j, k \dots$ cover the $\binom{n}{\lambda}$ ways in which the different co-sets are successively reduced.

Now let A be a particular matrix of the representation and let it be partitioned similarly in rows and columns; first into sets of $f^1, \dots f^h$ rows and columns, taken in order, and then again sub-divided into sets of $f^{11}, f^{12} \dots\dots f^{th}$ rows and columns - any numbers not otherwise defined being counted as zero. Let A denote the $f^r \times f^s$ matrix corresponding to the row set f^r and the column set f^s, and $A_{ri;sj}$ the $f^{ri} \times f^{sj}$ sub-matrix of this corresponding similarly to sets f^{ri}, f^{sj}.

Then if A corresponds to a substitution of $\mathfrak{h}_{[n-1]}$, application of this substitution to the ideal basis shows that

$$A = A_{11} \dotplus A_{22} \dotplus A_{33} \dotplus \dots\dots \dotplus A_{hh}.$$

Since the matrices corresponding to substitutions of lower groups can be obtained in this manner, an explicit form for a single inversion involving n will be sufficient to provide a set from which all others can be derived by matrix multiplication.

Selecting the inversion $(n, n-1)$ let its matrix be denoted by B and let it be partitioned in the above manner. It could be derived directly from the ideal basis but it is more convenient to make use of its commutative properties before seeking an expression for $(n, n-1)\, E_{rs}$ in terms of the basis. In fact, since $(n, n-1)$ commutes with all the substitutions of $\mathfrak{h}_{[n-2]}$ it follows that if A be one of these

$$A_{ri;ri} \cdot B_{ri,sj} = B_{ri;sj} \cdot A_{sj;sj}$$

Correspondingly in $\mathcal{O}_{n-2}$ quantities of $\mathfrak{A}_{[\lambda^{ri}]}$ are expressed in terms of those of $\mathfrak{A}_{[\lambda^{sj}]}$, and it follows that either $[\lambda^{ri}] = [\lambda^{sj}]$ and $f^{ri} = f^{sj}$ or $B_{ri;sj} = 0$. In the former case $B_{ri;sj}$ must be a multiple of the corresponding unit matrix: let the constant involved be denoted by $c_{ri;sj}$

Since $[\lambda^{ri}] = [\lambda^{sj}]$ implies either that $r = s$ and $i = j$ or that $r = j$ and $i = s$ it follows that, in the ideal basis, $(n, n-1)\, E_{rs}$ must be expressed in terms of E_{rs} and the corresponding E_{sr}. Suppose now that $r > s$, so that E_{sr} does not involve M_r. Then for inter-expressibility it is necessary that

$$e^{s+i,r}_{r,s} = 0 \qquad i = 1, 2 \dots .$$

since, in pre-multiplication with $(n, n-1)$, terms of M_{ij} go over into terms of M_{ji}. Hence in the summations for M_{sr} all terms are zero except those belonging to M_{rs}, and here precisely those correspond-:ing themselves to an $(n, n-1)$ inversion. Hence any particular $K_{\nu sr}$ is simply $c^s_r K_{\nu rs}$, so that

$$c_{rs;rs} = c^s_r = -\frac{1}{\lambda_s - \lambda_r + r - s}$$

This result remains valid, when $r < s$, since B is orthogonal and /

and its own inverse, so that

$$c_{rs;sr} = c_{sr;rs} = \frac{\sqrt{(\lambda_s - \lambda_r + r - s)^2 - 1}}{|\lambda_s - \lambda_r + r - s|}$$

and the orthogonality must be accounted for in the diagonal terms.

The fact that the square root carries a positive sign follows by considering $(n, n-1)\ E_{sr}$ in which quantities of M_{rs} must be accounted for by E_{rs} alone. Now if the e^i_i and corresponding lower forms are always taken over directly from lower ones in the isomorphism, the co-efficient corresponding to the latest $M_{rst\ldots}$ in any basis quantity will be a unit, divided by the corresponding positive normalising factors. Hence the multiple given to E_{rs} in the expression for $(n, n-1)$ must be positive.

It remains only to consider the co-efficients corresponding to

$$(n, n-1)\ E_{rr} = E_{rr} \quad ,$$

since the quantities e^{rr}_{rr} remain unaltered, and the co-efficients $c_{rs;rs}$ for which the corresponding $c_{sr;sr}$ are undefined. From the above relation one obtains immediately

$$c_{rr;rr} = 1 \quad ,$$

while in the E_{rs} for which E_{sr} does not exist the situation arises either because $\lambda_s = \lambda_{s+1}$, so that E_s does not exist, or because $\lambda^s_r = \lambda^s_{r+1}$. In the former case the existence of E_{rs} requires $r = s+1$, and the latter falls away. The argument used in the non-degenerate case still applies to e^{sr}_{rs}, and, substituting for s, λ_s in terms of r, λ_r we find

$$c_{rs;rs} = -1$$

Comparison of these results with the orthogonal form given by A. Young: "On quantitative substitutional Analysis (sixth paper)", Proc. London Math. Soc. (2), 34 (1931) p. 218, shows that the two matrix representations are identical.

SOME SIMPLE EXAMPLES OF THE IDEAL BASES.

The different values of $[\lambda]$ for $n = 1, 2 \ldots 6$ may be written in order as follows:

n = 1 $[1]$

n = 2 $[2]\ [1^2]$

n = 3 $[3]\ [21]\ [1^3]$

n = 4 $[4]\ [31]\ [2^2]\ [21^2]\ [1^4]$

n = 5 $[5]\ [41]\ [32]\ [31^2]\ [2^2 1]\ [21^3]\ [1^5]$

n = 6 $[6]\ [51]\ [42]\ [41^2]\ [3^2]\ [321]\ [31^3]\ [2^3]\ [2^2 1^2]\ [21^4]\ [1^5]$

Among those of the last row there appear some of rather large order, the $[2, 1^4]$, for example, involving 360 co-sets. In this paragraph the simple forms will be discussed, including all cases $n = 1 \ldots 5$ and also the $[3,2,1]$ of 6, this being the first to exhibit the method of construction in its full generality.

In the forms given below the quantities $E_1 \ldots E_r$ are expressed in a normalised form, in a tabular arrangement each column of which contains the co-efficients K_ν of $E_1 \ldots E_r$. The corresponding T_ν is represented at the head of the column in a tableau arrangement with μ_{ij}, $j = 1, 2 \ldots \lambda_i$ written in their natural order in the i^{th} column.

A second reduction into sets E_{rs} and M_{ij} is indicated and the full lattice permutation $rst \ldots$ is written on the right of each row. The corresponding values for the $(n, n-1)$ inversions are also given.

i. $\mathfrak{A}_{[n]}$:

There is only one co-set, corresponding to $E_{1^n} = 1 . T_\lambda$

ii. $\mathfrak{A}_{[n-1,1]}$:

There are n co-sets and the sum over these is $S_{[n]}$ corresponding to $[n]$, the only earlier partition. $\mathfrak{A}_{[n-1,1]}$ appears as a sum of $\mathfrak{A}_{[n-2,1]}$ and the unit representation $\mathfrak{A}_{[n-1]}$ The form assumed by matrices when the normalising factor is omitted is worth noting as it appears as the most obvious pair is the more complicated constructions.

ii. :

$$
\begin{array}{c|c|c}
 & \begin{array}{cccccccc} \substack{2\ 1\\3\\4\\\vdots} & \substack{1\ 2\\3\\4\\\vdots} & \substack{1\ 3\\2\\4\\\vdots} & \substack{1\ 4\\2\\3\\\vdots} & \substack{1\ 5\\2\\3\\\vdots} & \cdots & \substack{1\ n-1\\2\\3\\\vdots} & \substack{1\ n\\2\\3\\\vdots} \end{array} & \\
\hline
\begin{array}{c} E_{11} \\ \\ \\ \\ E_{12} \\ E_{21} \end{array} &
\left[\begin{array}{cccccccc}
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & & & & & \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & & & & & \\
-\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \sqrt{\frac{3}{4}} & & & & \\
-\frac{1}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & \sqrt{\frac{4}{5}} & & & \\
-\frac{1}{\sqrt{(n-1)(n-2)}} & -\frac{1}{\sqrt{(n-1)(n-2)}} & -\frac{1}{\sqrt{(n-1)(n-2)}} & -\frac{1}{\sqrt{(n-1)(n-2)}} & -\frac{1}{\sqrt{(n-1)(n-2)}} & & \sqrt{\frac{n-2}{n-1}} & \\
-\frac{1}{\sqrt{n(n-1)}} & -\frac{1}{\sqrt{n(n-1)}} & -\frac{1}{\sqrt{n(n-1)}} & -\frac{1}{\sqrt{n(n-1)}} & -\frac{1}{\sqrt{n(n-1)}} & & -\frac{1}{\sqrt{n(n-1)}} & \sqrt{\frac{n-1}{n}}
\end{array}\right] &
\begin{array}{c} 111\cdots1121 \\ 111\cdots1211 \\ 111\cdots2111 \\ 111\cdots21111 \\ 121\cdots111 \\ 211\cdots111 \end{array}
\end{array}
$$

$$
(n, n-1) \longrightarrow
\left[\begin{array}{cccccc}
1 & & & & & \\
 & 1 & & & & \\
 & & 1 & & & \\
 & & & 1 & & \\
 & & & & \ddots & \\
 & & & & & \begin{array}{cc} \frac{1}{n-1} & \frac{\sqrt{n(n-2)}}{n-1} \\ \frac{\sqrt{n(n-2)}}{n-1} & -\frac{1}{n-1} \end{array}
\end{array}\right]
$$

iii. $\mathfrak{A}_{[1^n]}$:

There are here $n!$ co-sets and $E_{n,n-1,\cdots 3,2,1} = \mathfrak{A}_n - (12)\mathfrak{A}_n$

For an odd substitution the method of constructions sums over all later even ones, divides by this total number, and changes the sign, and the even are derived likewise from the odd.

iv. $\mathfrak{A}_{[2^2]}$:

$$
\begin{array}{c}
\begin{array}{cccccc}
\begin{smallmatrix}3&1\\4&2\end{smallmatrix} & \begin{smallmatrix}2&1\\4&3\end{smallmatrix} & \begin{smallmatrix}1&2\\4&3\end{smallmatrix} & \begin{smallmatrix}2&1\\3&4\end{smallmatrix} & \begin{smallmatrix}1&2\\3&4\end{smallmatrix} & \begin{smallmatrix}1&3\\2&4\end{smallmatrix}
\end{array} \\
\begin{array}{c} E_{21} \\ E_{22} \end{array}
\left[\begin{array}{c:cc|cc:c}
\cdot & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \cdot \\ \hdashline
\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{array}\right]
\begin{array}{c} 2121 \\ 2211 \end{array}
\end{array}
\qquad (34) \rightarrow \left[\begin{array}{c:c} -1 & \cdot \\ \hdashline \cdot & 1 \end{array}\right]
$$

v. $\mathfrak{A}_{[3,2]}$:

$$
\begin{array}{c}
\begin{array}{cccccccccc}
\begin{smallmatrix}3&1\\4&2\\5\end{smallmatrix} & \begin{smallmatrix}2&1\\4&3\\5\end{smallmatrix} & \begin{smallmatrix}1&2\\4&3\\5\end{smallmatrix} & \begin{smallmatrix}2&1\\3&4\\5\end{smallmatrix} & \begin{smallmatrix}1&2\\3&4\\5\end{smallmatrix} & \begin{smallmatrix}1&3\\2&4\\5\end{smallmatrix} & \begin{smallmatrix}2&1\\3&5\\4\end{smallmatrix} & \begin{smallmatrix}1&2\\3&5\\4\end{smallmatrix} & \begin{smallmatrix}1&3\\2&5\\4\end{smallmatrix} & \begin{smallmatrix}1&4\\2&5\\3\end{smallmatrix}
\end{array} \\
\begin{array}{c} E_{12} \\ \\ E_{21} \\ \\ E_{22} \end{array}
\left[\begin{array}{ccc:ccc|ccc:c}
 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & & & & & \\
\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & & & & \\ \hline
 & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \cdot & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & & \\
\frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \\ \hdashline
\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{array}\right]
\begin{array}{c} 12121 \\ 12211 \\ 21121 \\ 21211 \\ 22111 \end{array}
\end{array}
\qquad
\left[\begin{array}{cc:cc:c}
\frac{1}{2} & & \frac{\sqrt{3}}{2} & & \\
 & \frac{1}{2} & & \frac{\sqrt{3}}{2} & \\ \hdashline
\frac{\sqrt{3}}{2} & & -\frac{1}{2} & & \\
 & \frac{\sqrt{3}}{2} & & -\frac{1}{2} & \\ \hdashline
 & & & & 1
\end{array}\right]
$$

vi. $\mathcal{A}_{[2,1^2]}$:

$$
\begin{array}{c|cccccccccccc|c}
 & \begin{smallmatrix}3\,2\,1\\4\end{smallmatrix} & \begin{smallmatrix}3\,1\,2\\4\end{smallmatrix} & \begin{smallmatrix}2\,3\,1\\4\end{smallmatrix} & \begin{smallmatrix}1\,3\,2\\4\end{smallmatrix} & \begin{smallmatrix}2\,1\,3\\4\end{smallmatrix} & \begin{smallmatrix}1\,2\,3\\4\end{smallmatrix} & \begin{smallmatrix}2\,4\,1\\3\end{smallmatrix} & \begin{smallmatrix}1\,4\,2\\3\end{smallmatrix} & \begin{smallmatrix}1\,4\,3\\2\end{smallmatrix} & \begin{smallmatrix}2\,1\,4\\3\end{smallmatrix} & \begin{smallmatrix}1\,2\,4\\3\end{smallmatrix} & \begin{smallmatrix}1\,3\,4\\2\end{smallmatrix} & \\
\hline
E_{13} & -\frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & \frac{1}{\sqrt6} & & & & & & & 1321 \\
\hline
E_{31} & -\frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & -\frac{1}{4\sqrt3} & \frac{1}{4\sqrt3} & \frac{1}{4\sqrt3} & -\frac{1}{4\sqrt3} & \frac{\sqrt3}{4} & -\frac{\sqrt3}{4} & \cdot & -\frac{\sqrt3}{4} & \frac{\sqrt3}{4} & & 3121 \\
E_{32} & \cdot & \cdot & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & 3211
\end{array}
$$

$$
(34) \longrightarrow \left[\begin{array}{c|c|c}
\frac{1}{3} & \frac{2\sqrt2}{3} & \cdot \\
\hline
\frac{2\sqrt2}{3} & -\frac{1}{3} & \cdot \\
\hline
\cdot & \cdot & -1
\end{array}\right]
$$

vii. $\mathfrak{A}_{[3,1^2]}$

		321 4 5	312 4 5	231 4 5	132 4 5	213 4 5	123 4 5	241 3 5	142 3 5	143 2 5	214 3 5	124 3 5	134 2 5	251 3 4	152 3 4	153 2 4	154 2 3	215 3 4	125 3 4	135 2 4	145 2 3	
E_{11}		$-\frac{1}{\sqrt6}$	$\frac{1}{\sqrt6}$	$\frac{1}{\sqrt6}$	$-\frac{1}{\sqrt6}$	$-\frac{1}{\sqrt6}$	$\frac{1}{\sqrt6}$															11321
E_{13}		$-\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$-\frac{1}{4\sqrt3}$	$\frac{1}{4\sqrt3}$	$\frac{1}{4\sqrt3}$	$-\frac{1}{4\sqrt3}$	$\frac{\sqrt3}{4}$	$-\frac{\sqrt3}{4}$		$-\frac{\sqrt3}{4}$	$\frac{\sqrt3}{4}$										13121
		·	·	$-\frac14$	$-\frac14$	$\frac14$	$\frac14$	$\frac14$	$\frac14$	$-\frac12$	$-\frac14$	$-\frac14$	$\frac12$									13211
E_{31}		$-\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{15}}$	$-\frac{1}{4\sqrt{15}}$	$\frac{1}{4\sqrt{15}}$	$\frac{1}{4\sqrt{15}}$	$-\frac{1}{4\sqrt{15}}$	$-\frac{1}{4\sqrt{15}}$	$\frac{1}{4\sqrt{15}}$	·	$\frac{1}{4\sqrt{15}}$	$-\frac{1}{4\sqrt{15}}$	·	$\frac{1}{\sqrt5}$	$-\frac{1}{\sqrt5}$	·	·	$-\frac{1}{\sqrt5}$	$\frac{1}{\sqrt5}$			31121
		·	·	$-\frac14\sqrt{\frac35}$	$-\frac14\sqrt{\frac35}$	$\frac14\sqrt{\frac35}$	$\frac14\sqrt{\frac35}$	$-\frac{1}{4\sqrt{15}}$	$-\frac{1}{4\sqrt{15}}$	$\frac{1}{2\sqrt{15}}$	$\frac{1}{4\sqrt{15}}$	$\frac{1}{4\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{\sqrt{15}}$	$\frac{1}{\sqrt{15}}$	$-\frac{1}{\sqrt{15}}$	·	$-\frac{1}{\sqrt{15}}$	$-\frac{1}{\sqrt{15}}$	$\frac{2}{\sqrt{15}}$		31211
E_{32}		·	·	·	·	·	·	$-\frac{1}{\sqrt{30}}$	$-\frac{1}{\sqrt{30}}$	$-\frac{1}{\sqrt{30}}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{\sqrt{30}}$	$-\sqrt{\frac{3}{10}}$	$-\frac{1}{\sqrt{30}}$	$-\frac{1}{\sqrt{30}}$	$-\frac{1}{\sqrt{30}}$	$\sqrt{\frac{3}{10}}$	32111

$$(45) \longrightarrow \begin{bmatrix} 1 & & & & & \\ & \frac14 & & \frac{\sqrt{15}}{4} & & \\ & & \frac14 & & \frac{\sqrt{15}}{4} & \\ & \frac{\sqrt{15}}{4} & & -\frac14 & & \\ & & \frac{\sqrt{15}}{4} & & -\frac14 & \\ & & & & & -1 \end{bmatrix}$$

viii $\mathfrak{A}_{[2^21]}$:

	421 53	412 53	413 52	321 54	312 54	231 54	132 54	213 54	123 54	314 52
21321	·	·	·	$\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	·
23121	$\frac{1}{4}\sqrt{\frac{3}{2}}$	$-\frac{1}{4}\sqrt{\frac{3}{2}}$	·	$-\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$-\frac{1}{2\sqrt6}$	$\frac{1}{2\sqrt6}$	$-\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	
23111	$\frac{1}{4\sqrt2}$	$\frac{1}{4\sqrt2}$	$-\frac{1}{2\sqrt2}$	$-\frac{1}{4\sqrt2}$	$-\frac{1}{4\sqrt2}$	·	·	$\frac{1}{4\sqrt2}$	$\frac{1}{4\sqrt2}$	$\frac{1}{2\sqrt2}$
32121	$\frac{1}{4\sqrt2}$	$-\frac{1}{4\sqrt2}$		$-\frac{1}{4\sqrt2}$	$\frac{1}{4\sqrt2}$	·	·	$\frac{1}{4\sqrt2}$	$-\frac{1}{4\sqrt2}$	·
32211	$\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$-\frac{1}{2\sqrt6}$	$\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$-\frac{1}{2\sqrt6}$	$-\frac{1}{2\sqrt6}$	$\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$-\frac{1}{2\sqrt6}$

	214 53	124 53	321 45	312 45	231 45	132 45	213 45	123 45	241 35	142 35
21321	·	·	$-\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$		
23121	$\frac{1}{4}\sqrt{\frac{3}{2}}$	$-\frac{1}{4}\sqrt{\frac{3}{2}}$	$-\frac{1}{2\sqrt6}$	$\frac{1}{2\sqrt6}$	$-\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$-\frac{1}{4\sqrt6}$	$\frac{1}{4}\sqrt{\frac{3}{2}}$	$-\frac{1}{4}\sqrt{\frac{3}{2}}$
23111	$-\frac{1}{4\sqrt2}$	$-\frac{1}{4\sqrt2}$	·	·	$-\frac{1}{4\sqrt2}$	$-\frac{1}{4\sqrt2}$	$\frac{1}{4\sqrt2}$	$\frac{1}{4\sqrt2}$	$\frac{1}{4\sqrt2}$	$\frac{1}{4\sqrt2}$
32121	$-\frac{1}{4\sqrt2}$	$\frac{1}{4\sqrt2}$	·	·	$-\frac{1}{4\sqrt2}$	$\frac{1}{4\sqrt2}$	$-\frac{1}{4\sqrt2}$	$\frac{1}{4\sqrt2}$	$\frac{1}{4\sqrt2}$	$-\frac{1}{4\sqrt2}$
32211	$\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$-\frac{1}{2\sqrt6}$	$-\frac{1}{2\sqrt6}$	$\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$

	143 25	214 35	124 35	134 25	315 42	215 43	125 43	215 34	125 34	135 24
21321										
23121		$-\frac{1}{4}\sqrt{\frac{3}{2}}$	$\frac{1}{4}\sqrt{\frac{3}{2}}$							
23111	$-\frac{1}{2\sqrt2}$	$-\frac{1}{4\sqrt2}$	$-\frac{1}{4\sqrt2}$	$\frac{1}{2\sqrt2}$						
32121	·	$\frac{1}{4\sqrt2}$	$-\frac{1}{4\sqrt2}$	·	·	$\frac{1}{2\sqrt2}$	$-\frac{1}{2\sqrt2}$	$-\frac{1}{2\sqrt2}$	$\frac{1}{2\sqrt2}$	
32211	$-\frac{1}{2\sqrt6}$	$\frac{1}{4\sqrt6}$	$\frac{1}{4\sqrt6}$	$-\frac{1}{2\sqrt6}$	$\frac{1}{\sqrt6}$	$-\frac{1}{2\sqrt6}$	$-\frac{1}{2\sqrt6}$	$-\frac{1}{2\sqrt6}$	$-\frac{1}{2\sqrt6}$	$\frac{1}{\sqrt6}$

$$(45) \longrightarrow \left[\begin{array}{c|cc|cc} -1 & & & & \\ \hline & \frac{1}{2} & & \frac{\sqrt3}{2} & \\ & & \frac{1}{2} & & \frac{\sqrt3}{2} \\ \hline & \frac{\sqrt3}{2} & & -\frac{1}{2} & \\ & & \frac{\sqrt3}{2} & & -\frac{1}{2} \end{array}\right]$$

ii. The reductions corresponding to partitions $[321]$ and $[21^3]$ are exhibited on a separate sheet. The examples given above have been written in the proper normalised form required in the construction of orthogonal matrices to reduce the regular representation or in generating the group matrices. This presentation has, however, the disadvantage of obscuring the various isomorphisms relating to restricted operator domains, since the normalising factors are not in general the same for the isomorphic sets. For this reason the two now given are written in an integral form obtained by reversing the signs on continuation into earlier columns, and by multiplying all later ones by the corresponding factor, $\frac{1}{c_r^t}$. The single normalising factor required to reduce each row to its proper normalised form is written on the left of the matrix.

It will have been observed that, in the preceding example, some of the earlier isomorphic sets go over into a reversed form of the corresponding reduced matrices. This is due to a defect in the natural ordering of co-sets adhered to throughout and arises in the cases where successive reduction of co-sets leads to partitions of lower order with their columns no longer taken in the order corresponding to their lengths. The original order used below is the same as before, it being sufficient to notice why sets of the type

$$
\begin{array}{r}
1 - 1 \\
2 - 1 - 1 \\
3 - 1 - 1 - 1
\end{array}
$$

should arise in the construction.

$2\sqrt{3}$	·	·	·	1	-1	-1	1	1	-1	·	·	·	-1	1	1	-1	-1	1																																										
$4\sqrt{6}$	3	-3	·	-1	1	-2	2	-1	1	·	3	-3	-2	2	-1	1	1	-1	3	-3	·	-3	3	·																																				
$4\sqrt{2}$	-1	1	-2	-1	-1	·	·	1	1	2	-1	-1	·	·	-1	-1	1	1	1	1	-2	-1	-1	2																																				
$4\sqrt{2}$	1	-1	·	-1	1	·	·	1	-1	·	-1	1	·	·	-1	1	-1	1	1	-1	·	1	-1	·	·	2	-2	-2	2	·																														
$4\sqrt{6}$	1	1	-2	1	1	-2	-2	1	1	-2	1	1	-2	-2	1	1	1	1	1	1	-2	1	1	-2	4	-2	-2	-2	-2	4																														
$6\sqrt{3}$	·	·	·	1	-1	-1	1	1	-1	·	·	·	1	-1	-1	1	1	-1	·	·	·	·	·	·	·	·	·	·	·	·	-4	4	4	-4	-4	4																								
$12\sqrt{6}$	3	-3	·	-1	1	-2	2	-1	1	·	3	-3	2	-2	1	-1	-1	1	-3	3	·	3	-3	·	·	·	·	·	·	·	-8	8	-4	4	4	-4	12	-12	·	-12	12	·																		
$36\sqrt{2}$	3	3	-6	-3	-3	·	·	3	3	6	-3	-3	·	·	3	3	-3	-3	-3	-3	6	3	3	-6	·	·	·	·	·	·	·	·	-12	-12	12	12	12	12	-24	-12	-12	24																		
$12\sqrt{10}$	3	-3	·	3	-3	2	-2	-1	1	·	-1	1	-2	2	-3	3	-1	1	-3	3	·	-1	1	·	·	4	-4	4	-4	·	-8	8	-4	4	4	-4	-4	4	·	4	-4	·	16	-16	·	·	-16	16												
$12\sqrt{30}$	3	3	-6	1	1	4	4	-5	-5	-2	1	1	-4	-4	-1	-1	5	5	-3	-3	6	-1	-1	2	8	-4	-4	4	4	-8	·	·	-12	-12	12	12	-4	-4	8	4	4	-8	16	16	-32	·	-16	-16	32	·										
$24\sqrt{15}$	·	·	·	4	4	4	4	4	4	-8	-8	-8	-4	-4	-4	-4	-4	-4	·	·	·	8	8	8	8	8	8	-8	-8	-8	·	·	·	·	·	·	-16	-16	-16	16	16	16	16	16	16	-48	-16	-16	-16	48										
$4\sqrt{30}$	3	-3	·	-3	3	·	·	3	-3	·	-3	3	·	·	1	-1	1	-1	-1	1	·	-1	1	·	·	-2	2	2	-2	·	·	·	-4	4	-4	4	4	-4	·	4	-4	·	·	·	·	·	·	·	·	·	·	8	-8	-8	8					
$12\sqrt{10}$	3	3	-6	3	3	-6	-6	3	3	-6	3	3	2	2	-1	-1	-1	-1	-1	-1	2	-1	-1	2	-4	2	2	2	2	-4	-8	-8	4	4	4	4	4	4	-8	4	4	-8	·	·	·	·	·	·	·	·	16	-8	-8	-8	-8	16				
$12\sqrt{10}$	3	-3	·	3	-3	2	-2	-1	1	·	-1	1	-6	6	-1	1	5	-5	-1	1	·	5	-5	·	·	-4	4	-4	4	·	·	·	-4	4	-4	4	-4	4	·	-4	4	·	8	-8	·	·	8	-8	·	·	·	8	-8	8	-8	·	-16	16		
$12\sqrt{30}$	3	3	-6	1	1	4	4	-5	-5	-2	1	1	4	4	-11	-11	7	7	-1	-1	2	5	5	-10	-8	4	4	-4	-4	8	-8	-8	4	4	4	4	-4	-4	8	-4	-4	8	8	8	-16	·	8	8	-16	·	16	-8	-8	8	8	-16	-16	-16	32	
$24\sqrt{15}$	·	·	·	4	4	4	4	4	4	-8	-8	-8	4	4	4	4	4	4	-16	-16	-16	8	8	8	-8	-8	-8	8	8	8	-8	-8	-8	-8	-8	-8	8	8	8	8	8	8	8	8	8	-24	8	8	8	-24	16	16	16	-16	-16	-16	-16	-16	-16	48

$2\sqrt{6}$	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1																																				
$6\sqrt{10}$	3	-3	-3	3	3	-3	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-4	4	4	-4	-4	4	·	·	·	·	·	·	4	-4	-4	4	4	-4	·	·	·	·	·	·	-4	4	4	-4	-4	4						
$24\sqrt{5}$	·	·	·	·	·	·	8	-8	4	-4	-4	4	-8	8	-4	4	4	-4	8	-8	4	-4	-4	4	-8	8	-4	4	4	-4	12	-12	·	-12	12	·	8	-8	4	-4	-4	4	-12	12	·	12	-12	·	-8	8	-4	4	4	-4	12	-12	·	-12	12	
$8\sqrt{15}$	·	·	·	·	·	·	·	·	4	4	-4	-4	·	·	-4	-4	4	4	·	·	4	4	-4	-4	·	·	-4	-4	4	4	4	4	-8	-4	-4	8	·	·	4	4	-4	-4	-4	-4	8	4	4	-8	·	·	-4	-4	4	4	4	4	-8	-4	-4	8

8. CONJUGACY WITH RESPECT TO $\mathfrak{A}_n$

It was shown in paragraph 2 that, given any left ideal in $\mathcal{O}$ one can construct a corresponding conjugate ideal and generate from the two conjugate representations, these standing in a reciprocal relation to each other. The conjugate representation can be obtained in a form which differs from the original only in having an additional factor -1 applied to the odd permutations. We propose to derive here from the results of paragraph 6 the conditions under which the two are equivalent, and to investigate in other cases the forms of the associated partitions.

In the canonical form for $\mathfrak{A}_{[\lambda]}$ it is easily seen that, if there are ν_j distinct λ_{ij}, $i = 1, 2 \ldots\ldots$ then the earliest quantity in the basis corresponds to the direct lattice permutation.

$$1\,2 \ldots\ldots \nu_1\,1\,2 \ldots\ldots \nu_2\,1\,2 \ldots\ldots \nu_{\lambda_1}$$

where the word 'direct' is used to distinguish between this and the reversed form used in the notation $E_{rst\ldots\ldots}$. Since all the continuations involved in deriving this basis element involve only co-efficients ± 1, with the exception of the final orthogonalising factor, it is clear, by an extension of the argument applied to the $[1^n]$ representation in the preceding section, that the basis quantity differs from the Young operator NP in which P is a column operator, N a row operator, only in the constant orthogonal:ising factor. If $[\lambda^*]$ denotes the partition conjugate to $[\lambda]$ in the ordinary sense, and if the τ_μ expressing the co-set $\mathcal{T}_\mu$ which characterises the basis quantity is taken with the columns in order, N represents $\mathcal{T}^*_{[\lambda^*]}$ and P $\tau_\mu\, \mathcal{T}_{[\lambda]}$. Hence

$$\mathcal{T}^*_{[\lambda^*]} \cdot \tau_\mu \cdot \mathcal{T}_{[\lambda]} \;\varepsilon\; \mathfrak{A}_{[\lambda]}$$

and similarly

$$\mathcal{T}^*_{[\lambda]} \cdot \tau_\mu^{-1} \mathcal{T}_{[\lambda^*]} \;\varepsilon\; \mathfrak{A}_{[\lambda^*]},$$

by the reciprocal nature of the partition conjugacy and the substitutions τ_μ, τ_μ^{-1}. But these relations express the fact that the intersection of $\overline{\mathfrak{A}}_{[\lambda]}{}^*$ and $\mathfrak{A}_{[\lambda^*]}$ is not zero, and the irreducibility then shows that they are actually identical when extended to two sided ideals.

It follows from this argument that, when two-sided ideals are under consideration, the process of taking conjugates in $\mathfrak{A}_n$ is equivalent to that of passing to conjugate partitions. In particular for left ideals there is an isomorphism between irreducible ideals of types corresponding to conjugate partitions, and for matric representations one can pass from one to the other by /

by a change of sign as described above. If the partition is self conjugate the two representations thus obtained can be transformed into each other. It follows in particular that in this case the characters of odd substitutions are zero.

9. THE COMPLETE REDUCTION OF THE IDEALS $(S_{[\mu]})$.

It was shown in paragraph 7 that the left ideals $(S_{[\mu]})$ break up into two parts; the irreducible $\mathfrak{A}_{[\mu]}$ and a reciprocal ideal $\mathcal{N}_{[\mu]}$ which must be expressible as a sum of ideals corres-:ponding to earlier partitions $[\lambda]$. The importance of determining exactly which partitions $[\lambda]$ appear in any of these ideals, and of finding a basis for them in terms of co-sets of type $[\mu]$ will be explained in later sections. We proceed first to perform the complete reduction.

Since the set of all $e_i \bar{e}_j$ taken over all partitions $[\lambda]$ of n forms a complete basis for $\mathcal{O}$, the left ideal generated by $S_{[\mu]}$ consists of combinations of all non-zero quantities of the form

$$e_i\, \bar{e}_j \cdot S_{[\mu]} \qquad , \qquad e_i \in \mathfrak{A}_{[\lambda]}\, ,$$

taken over all values of $[\lambda]$. Moreover, by the orthogonality and irreducibility, this reduction expresses $(S_{[\mu]})$ as a direct sum of different ideals, each being zero or a sum of isomorphic irreducible ones. Keeping $[\lambda]$ and j fixed, the set taken over $i = 1, 2 \cdots f$ forms a left ideal, so that if one of these quantities is zero, all must be. In particular the non-zero ideals appearing in this reduction are specified completely when it has been determined how many of the quantities $e_f\, \bar{e}_j \cdot S_{[\mu]}$ are non-zero, and to what degree these overlap.

It will be more convenient to consider instead the inversely isomorphic quantities $S_{[\mu]} E_\sigma$ where σ will be used to denote the direct lattice permutation $\sigma_1 \cdots \sigma_n$. Let this be partitioned into sets $\sigma_{\mu_{ij}}$, $j = 1 \cdots \mu_i$, conforming with the partition $[\mu]$ of n. Then by applying to E_σ the substitutions of $S_{[\mu]}$ each expressed as a product of adjacent inversions, and by expressing the result of each such inversion in terms of the original basis before applying the next, it is obvious from the canonical form for these inversions, that $S_{[\mu]} E_\sigma$, expressed in terms of the original basis, assumes the form

$$\sum_\tau \kappa_\tau\, E_\tau \qquad \kappa_\tau \in K\, ,$$

in which the summation is restricted to lattice permutations which can be obtained from σ by re-arrangements within the sub-sets.

It is easily seen that the co-efficients κ_τ are either all zero or stand in a fixed ratio to each other. For the expression for $S_{[\mu]} \cdot E_\sigma$ is invariant under pre-multiplication by any substitution of the form $(\mu_{ij}\, \mu_{i,j+1})$, and this condition is expressed /

expressed by a relation between two constants K_τ, $K_{\tau'}$ for every significant permutation in the $\sigma_{\mu_{ij}}$ except the degenerate case when τ' does not exist and $K_\tau = -K_\tau = 0$. The relation has the form

$$\frac{K_\tau}{K_{\tau'}} = \sqrt{\frac{\lambda'_{\sigma_{ij}} - \lambda'_{\sigma_{i,j+1}} + \sigma_{i,j+1} - \sigma_{ij} - 1}{\lambda'_{\sigma_{ij}} - \lambda'_{\sigma_{i,j+1}} + \sigma_{i,j+1} - \sigma_{ij} + 1}}$$

unless $K_{\tau'} = 0$, in which case K_τ is also zero. The symbols used above are not essential to the argument and represent some reduced form of the partition. It is sufficient to know that the case is not degenerate.

If a permutation of σ yields a degenerate τ of the form considered above then K_τ is zero in the expansion of $S_{[\mu]} E_\sigma$ by the argument given above. But it is also true that $S_{[\mu]} E_\tau$ is zero, since $S_{[\mu]}$ contains a right hand factor of the type $\{1 + (\mu_{ij}\,\mu_{i,j+1})\}$ which, applied to E_τ, gives $E_\tau - E_\tau = 0$. It is however possible to pass from a σ with a degenerate permuted form $(\tau$ by a non-degenerate processes involving adjacent inversions in descending order, and so to obtain a series of relations between constants, of the type considered above, commencing with K_σ and terminating with the vanishing K_τ. Hence all the co-efficients vanish and $S_{[\mu]} \cdot E_\sigma = 0$.

There remains the case in which none of the permutations obtained from σ are degenerate. All the co-efficients are determined uniquely from any given one and all are non-zero. To see this it is sufficient to notice first that the relations between the co-efficients imply, in any case, that the quantities $S_{[\mu]} \cdot E_\tau$ corresponding to τ which are permuted forms of σ differ at most by a non-zero constant, and then to consider the case where τ is the earliest such form and arrange the permutations of $S_{[\mu]}$ as adjacent inversions, ascending from right to left, so that all the linear combinations formed when the result of multiplying by an inversion is referred to the original basis involves the earlier row of the corresponding matrix, so that the whole process of multiplication goes through with positive signs and cannot lead to zero.

It will be convenient to summarise these results by terming all lattice permutations obtained from a fixed σ in the above manner, equivalent with respect to $[\mu]$. Then if σ is equivalent to a degenerate permutation $S_{[\mu]} \cdot E_\sigma$ is zero, otherwise it is definitely non-zero and differs only by a constant for equivalent σ. Finally, for non-equivalent, non-degenerate σ, τ the quantities $S_{[\mu]} E_\sigma$, $S_{[\mu]} E_\tau$ are orthogonal, since they consist of combinations of mutually exclusive sets of orthogonal quantities. Let the number of such distinct products be denoted by f^μ_λ

Returning now to the original quantities

$$\bar{E}_\sigma \cdot S_{[\mu]} = \vartheta . E_{1^{\lambda_1} 2^{\lambda_2} \ldots} \quad \bar{E}_\sigma \cdot S_{[\mu]} \quad , \quad \vartheta \in K,$$

orthogonal/

orthogonal for non equivalent σ the isomorphism of the basis, together with the irreducibility, shows that for these σ all the $E_\tau \bar{E}_\sigma S_{[\mu]}$ are orthogonal in σ, while the inverse ring shows them orthogonal in τ.

Some examples of this type of reduction have already been given in paragraph 7 where the process of continuation which derives from sets e_ν^ν the isomorphic sets e_ν^j actually corresponds in the $\mathcal{O}_{n-1}$ isomorphism, to constructing ideals of type $[\lambda']$ as irreducible parts of $(S_{[\lambda']})$. The correspondence defined in paragraph 5 could have been expressed in terms of the results of this section and the construction completed by a full induction on n but as this would have involved an induction on the form of the inversions it has been considered more appropriate to obtain the basis for $\mathfrak{A}_{[\lambda]}$ in the first instance by a process which bears a closer relation to the practical construction, and to regard the present investigation as an explanation of the principles which lie behind the correspondence given there.

An illustration involving repeated ideals is given below, the left ideal $(S_{[2,1^2]})$ being expressed in terms of its simplest components.

$$
\begin{array}{c|cccccccccccc|}
 & \begin{smallmatrix}321\\4\end{smallmatrix} & \begin{smallmatrix}312\\4\end{smallmatrix} & \begin{smallmatrix}231\\4\end{smallmatrix} & \begin{smallmatrix}132\\4\end{smallmatrix} & \begin{smallmatrix}213\\4\end{smallmatrix} & \begin{smallmatrix}123\\4\end{smallmatrix} & \begin{smallmatrix}241\\3\end{smallmatrix} & \begin{smallmatrix}142\\3\end{smallmatrix} & \begin{smallmatrix}143\\2\end{smallmatrix} & \begin{smallmatrix}214\\3\end{smallmatrix} & \begin{smallmatrix}124\\3\end{smallmatrix} & \begin{smallmatrix}134\\2\end{smallmatrix} \\ \hline
 & -\frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\bar{E}_{1123} & -\frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & -\frac{1}{4\sqrt3} & \frac{1}{4\sqrt3} & \frac{1}{4\sqrt3} & -\frac{1}{4\sqrt3} & \frac{\sqrt3}{4} & -\frac{\sqrt3}{4} & \cdot & -\frac{\sqrt3}{4} & \frac{\sqrt3}{4} & \cdot \\
 & \cdot & \cdot & -\frac14 & -\frac14 & \frac14 & \frac14 & \frac14 & \frac14 & -\frac12 & -\frac14 & -\frac14 & \frac12 \\ \hline
\bar{E}_{1122} & \cdot & \cdot & \frac{1}{2\sqrt2} & -\frac{1}{2\sqrt2} & \frac{1}{2\sqrt2} & -\frac{1}{2\sqrt2} & -\frac{1}{2\sqrt2} & \frac{1}{2\sqrt2} & \cdot & -\frac{1}{2\sqrt2} & \frac{1}{2\sqrt2} & \cdot \\
 & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{1}{2\sqrt6} & -\frac{1}{2\sqrt6} & -\frac{1}{2\sqrt6} & -\frac{1}{2\sqrt6} & -\frac{1}{2\sqrt6} & -\frac{1}{2\sqrt6} & \frac{1}{\sqrt6} & -\frac{1}{2\sqrt6} & -\frac{1}{2\sqrt6} & \frac{1}{\sqrt6} \\ \hline
 & \frac{1}{2\sqrt3} & -\frac{1}{2\sqrt3} & -\frac{1}{4\sqrt3} & \frac{1}{4\sqrt3} & -\frac{\sqrt3}{4} & \frac{\sqrt3}{4} & -\frac{1}{4\sqrt3} & \frac{1}{4\sqrt3} & \cdot & -\frac{\sqrt3}{4} & \frac{\sqrt3}{4} & \cdot \\
\bar{E}_{1121} & -\frac13 & -\frac13 & \frac{5}{12} & \frac{5}{12} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & \frac16 & -\frac14 & -\frac14 & \frac12 \\
 & -\frac{1}{3\sqrt2} & -\frac{1}{3\sqrt2} & -\frac{1}{3\sqrt2} & -\frac{1}{3\sqrt2} & -\frac{1}{3\sqrt2} & -\frac{1}{3\sqrt2} & \frac{\sqrt2}{3} & \frac{\sqrt2}{3} & \frac{\sqrt2}{3} & \cdot & \cdot & \cdot \\ \hline
 & -\frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \cdot & \cdot & -\frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \cdot & \cdot & \cdot & \cdot \\
\bar{E}_{1112} & -\frac{1}{3\sqrt2} & -\frac{1}{3\sqrt2} & -\frac{1}{3\sqrt2} & -\frac{1}{3\sqrt2} & \frac{\sqrt2}{3} & \frac{\sqrt2}{3} & -\frac{1}{3\sqrt2} & -\frac{1}{3\sqrt2} & \frac{\sqrt2}{3} & \cdot & \cdot & \cdot \\
 & -\frac16 & -\frac16 & -\frac16 & -\frac16 & -\frac16 & -\frac16 & -\frac16 & -\frac16 & -\frac16 & \frac12 & \frac12 & \frac12 \\ \hline
\bar{E}_{1111} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3}
\end{array}
$$

In the case $[\mu] = [1^n]$ the method described above reverts to the original manner in which the reduction of the regular representation was described in paragraph 4, and does not give any new information on the form of other elements. Nevertheless it will perhaps not be out of place to describe here the complete reduction and the form which the matrices assume in case $n = 4$, this being the highest order which one can investigate in its full $n! \times n!$ form without the expressions becoming rather unwieldy. Unfortunately it is not a typical case because the $[2^2]$ representation gives no more information than the $[2,1]$ of case $n = 3$, since it makes use of the Klein sub-group as a normal divisor, as mentioned in paragraph 2.

The matrix exhibited below has been obtained by applying the forms obtained for $(n, n-1)$ inversions to the direct sum of representations of lower order. In contrast to the form already described for $n = 3$, the matrix is not orthogonal, as it has been considered more appropriate to exhibit it ~~the matrix~~ in a form which involves the ϑ_{ij} of paragraph 2, but which has the advantage of expressing directly the different elements of the group matrices. Accordingly, the order adopted for each representation runs down the columns in order so that the last f rows of the matrix contain the elements of the last column. These are written down directly from the ideal basis described above and exhibit the symmetry character-:istic of the particular representation. In completing the matrix account is of course taken of the orthogonality so that the elements of the last row, for example, can be written down directly.

To pass from the normalised ideal basis to the elements of the group matrix, the most convenient method is to normalise the rows separately, proceeding from the last. The fact that, in the canonical form, no two rows terminate with the same co-set, makes this a simple process.

In writing out the matrix some convenient ordering of the full set of substitutions is required. For any particular representation one would of course adopt the co-set order used above, but this would immediately destroy the symmetry of any of the other representations. It has therefore been considered appropriate to write the sets in an order which exhibits a very simple type of endomorphism applicable to the successively reduced operator domain.

This is obtained by writing each substitution in the form

$$(1K_1)(2K_2)\cdots\cdots(nK_n) = K_1K_2\cdots\cdots K_n \quad , \qquad K_i \leq i \, ,$$

so that the columns, which appear together in the matrix, break up into simple left ideals with each depression in the order of the operator domain, each of these being simply isomorphic with all others appearing in the same row, apart possibly from zero terms.

It might have been supposed that under this, or under some similar /

similar type of decomposition, the matrix would exhibit a structural form suggesting a simpler mode of definition than that given in paragraph 5. The example given here seems to indicate that if such a form exists it is not associated with the symmetry conditions typifying the different representations, or with the inductive form of construction in which each substitution of lower order is represented by a direct sum of earlier representations.

1111: (1432)	1211: (143)	1121: (1423)	1221: (14)(23)	1131: (142)	1231: (14)	1112: (1324)	1212: (13)(24)	1122: (1243)	1222: (243)	1132: (124)	1232: (24)	1113: (1342)	1213: (134)	1123: (1234)	1223: (234)	1133: (12)(34)	1233: (34)	1114: (132)	1214: (13)	1124: (123)	1224: (23)	1134: (12)	1234: 1	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1111
$-\frac{1}{2}$	$\frac{1}{2}$	.	.	$-\frac{1}{2}$	$\frac{1}{2}$	.	.	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	1	
$-\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$\frac{1}{\sqrt3}$	$\frac{1}{\sqrt3}$	$-\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$-\frac{1}{\sqrt3}$	$-\frac{1}{\sqrt3}$	$\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	.	.	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	.	.	
$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	.	.	.	.	.	.	.	.	.	.	.	.	
$\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$-\frac{1}{\sqrt3}$	$\frac{1}{\sqrt3}$	$\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$\frac{1}{\sqrt3}$	$-\frac{1}{\sqrt3}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$-\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	.	.	$\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	.	.	
$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	1	
$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	.	.	.	.	.	.	
.	.	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	.	.	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	.	.	.	.	.	.	.	.	1211
$\frac{2\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	.	.	.	.	.	.	1121
$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1	1	1	1	1	1	1112
$\frac{1}{2}$	$-\frac{1}{2}$	-1	1	$-\frac{1}{2}$	$\frac{1}{2}$	-1	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	1	
$-\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	.	.	$\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	.	.	$\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	.	.	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	.	.	
$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	.	.	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	.	.	$\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	.	.	$\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	.	.	1212
$-\frac{1}{2}$	$-\frac{1}{2}$	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	1	1122
$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	1	-1	1	-1	-1	1	
$\frac{2\sqrt2}{3}$	$-\frac{2\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$-\frac{2\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$-\frac{2\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	.	.	.	.	.	.	
.	.	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	.	.	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	.	.	.	.	.	.	.	.	
$-\frac{\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$\frac{\sqrt2}{3}$	$-\frac{\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$-\frac{2\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	$-\frac{2\sqrt2}{3}$	$-\frac{2\sqrt2}{3}$	$\frac{2\sqrt2}{3}$	.	.	.	.	.	.	
$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{5}{6}$	$-\frac{5}{6}$	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{5}{6}$	$-\frac{5}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	1	
$\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$-\frac{1}{\sqrt3}$	$-\frac{1}{\sqrt3}$	$-\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$\frac{1}{\sqrt3}$	$\frac{1}{\sqrt3}$	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	.	.	$\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	.	.	
$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	.	.	.	.	.	.	.	.	.	.	.	.	1231
$-\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$\frac{1}{\sqrt3}$	$-\frac{1}{\sqrt3}$	$\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$-\frac{1}{\sqrt3}$	$\frac{1}{\sqrt3}$	$\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}$	$\frac{1}{2\sqrt3}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	.	.	$\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt3}{2}$	$\frac{\sqrt3}{2}$	.	.	1213
$\frac{1}{2}$	$\frac{1}{2}$	.	.	$-\frac{1}{2}$	$-\frac{1}{2}$	.	.	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	1	1123
-1	1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	-1	1	1	-1	1	-1	1	-1	-1	1	1234

$$A_{[2,1^2]} = \pm \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix} A_{[3,1]} \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix}.$$

10. THE ORTHOGONAL REDUCTION OF TENSOR SPACE.

In attempting to carry out the full investigation outlined in the introduction, we proceed now to the investigation of relations which arise when homogeneous products of n^2 elements a_{ij} with repetitions in either or both sets of symbols are admitted. Consider an orthogonal matrix

$$A = [a_{ij}]$$

representing a transformation in an n-dimensional space. This induces a transformation

$$A^{\{n\}} = [a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}]$$

on an arbitrary tensor of rank n. In this $i_1 i_2 \cdots i_n$; $j_1 j_2 \cdots j_n$, representing the rows and columns respectively, run through all values obtained by permuting the symbols and allowing repetitions. The discussion will be restricted to the case when the rank n is the same as the order of the original matrix. If A is orthogonal so also is $A^{\{n\}}$ and the correspondence

$$A \longrightarrow A^{\{n\}}$$

defines an orthogonal representation of the group of matrices A by matrices in the tensor space. The present section will be devoted to the problem of reducing $A^{\{n\}}$ to a direct sum of irreducible components each of which will give an irreducible orthogonal representation of the matrices in tensor space.

To accomplish such a reduction we return to the strict distinction between modul and ring observed in earlier sections, and to the type of symbolisation which in paragraph 4, related the problem of constructing sub-moduls invariant under the operator domain to that of reducing the group matrix. Obviously the type of symbolism required here to represent the combinations of rows and columns of $A^{\{n\}}$ formed in orthogonal transformation will be obtained by writing

$$a_{i_1 j_1} \cdots\cdots a_{i_n j_n} = \alpha_{1 i_1} \cdots\cdots \alpha_{n i_n} \cdot \alpha_{1 j_1} \cdots\cdots \alpha_{n j_n}$$

in which the symbols are allowed to commute, the particular α associated with any α in a product being fixed by the first suffix. It should be observed that the application of the same permutation to the two sets of first suffices does not alter the result. In fact /

fact $a_{1i_1}\alpha_{i_1 j}, a_{2i_1}\alpha_{2j}, \ldots\ldots a_{ni_n}\alpha_{nj}$ can be regarded as equivalent symbols for $a_{i,j}$ in much the same way as these arise in invariant theory.

As explained in paragraph 4, any pre-multiplying matrix will form linear combinations of the sets $a_{1i_1} \ldots a_{ni_n}$ and any post-multiplier similar combinations of column symbols. In particular under orthogonal transformations the two sets will be identical in form. Extending the notations of earlier sections, let $\mathcal{M}$ denote the modul formed from all linear combinations of the row symbols with co-efficients in K, and $\overline{\mathcal{M}}$ the corresponding modul for column symbols.

Any row or column symbol can be associated with a definite partition $[\mu]$ of n in which μ_1 represents the greatest number of equal i_r in the symbol, μ_2 the next, and so on, repetitions of equal length being counted as often as they appear. Correspondingly $\mathcal{M}$ and $\overline{\mathcal{M}}$ may be broken up into sub-moduls $\mathcal{M}_\mu, \overline{\mathcal{M}}_\mu$ containing all combinations of symbols of type $[\mu]$. Then any quantities $Q \in \mathcal{M}$, $\overline{R} \in \overline{\mathcal{M}}$ respectively, can be expressed uniquely as sums:

$$Q = \sum q_\mu, \quad \overline{R} = \sum \overline{r}_\mu \qquad q_\mu \in \mathcal{M}_\mu, \; \overline{r}_\mu \in \overline{\mathcal{M}}_\mu .$$

In particular there is a sub-modul $\mathcal{M}_{[1^n]}$ corresponding exactly to the modul $\mathfrak{M}$ considered in the group representation, and the process of combining symbols $q_{[1^n]}$ and $\overline{r}_{[1^n]}$, regarded as an operation on $\mathcal{M}_{[1^n]}$, is isomorphic with the endomorphism ring of $\mathfrak{M}$. In the more general case of combinations between $q_{[\mu]}$ and $\overline{r}_{[\nu]}$ it is sufficient to observe that for each different value of $[\mu]$ or $[\nu]$ a different set of quantities are obtained. This can be expressed by saying that the sets $\mathcal{M}_\mu \overline{\mathcal{M}}_\nu$ do not overlap.

To reduce $A^{\{n\}}$ for any A it is necessary to construct combinations of symbols falling into orthogonal sets and satisfying relations of the type

$$Q.\overline{R} = 0; \qquad Q.\overline{S} = 0$$

where Q,R,S ...... are representatives of different sets. But the first relation implies

$$q_\mu \cdot \overline{r}_\nu = 0$$

and this cannot be satisfied by any non-zero q_μ if $\overline{r}_{[1^n]}, \overline{s}_{[1^n]} \ldots$ form a complete basis for $\overline{\mathcal{M}}_{[1^n]}$, since such a relation would involve the linear dependence of essentially distinct elements of $A^{\{n\}}$. It follows that, for some Q, $q_{[1^n]} \neq 0$. But this $q_{[1^n]}$ must now satisfy

$$q_{[1^n]}\overline{r}_{[1^n]} = 0; \qquad q_{[1^n]}\overline{S}_{[1^n]} = 0 , \quad \ldots\ldots$$

and hence the set of all $q_{[1^n]}$ goes over in the isomorphism to an ideal in $\mathcal{O}$.

The argument given above shows that if it is possible to reduce $A^{\{n\}}$, then each part in the reduction involves a part of $\mathfrak{M}_{[1^n]}$ isomorphic with an ideal of $\mathcal{O}$ so that the reduction certainly cannot go further than one into $\sum f_\mu$ irreducible parts. It will now be shown that the reduction does actually proceed this far. The irreducibility will then be implied.

Let $q_{[1^n]}$ be any quantity of $\mathfrak{M}_{[1^n]}$ corresponding to a quantity in an irreducible ideal of $\mathcal{O}$ and let $r_{[1^n]}$ correspond similarly to one in the reciprocal ideal. Then

$$q_{[1^n]} \cdot \bar{r}_{[1^n]} = 0$$

for all q, r of the type considered. Now divide each of the $\mathfrak{M}_\mu$ into a number of sub-sets each of which contains repetitions of type $[\mu]$ but differs from others in having an essentially distinct selection of repeated symbols. Let $\mathfrak{M}_\mu^i$ denote a typical sub-set.

It is now possible to define a series of correspondences in which each $q_{[1^n]}$ expresses a unique $q_\mu^i \in \mathfrak{M}_\mu^i$. For in any $\mathfrak{M}_\mu^i$ with $[\mu] \neq [1^n]$ there appear only h of the symbols $1, 2, \ldots n$ and one might associate, by any definite prescribed rule, $\mu_1 - 1$ of the missing symbols with that associated with μ_1; $\mu_2 - 1$ with that associated with μ_2, and so on until all the symbols and partition parts are exhausted. Then from any $q_{[1^n]}$ one can derive a quantity of $\mathfrak{M}_\mu^i$ by giving to each of the μ_1 symbols appearing in $q_{[1^n]}$ the single value appropriate to $\mathfrak{M}_\mu^i$ and treating other sets similarly. Let the result of this operation be denoted by q_μ^i and the process by which it was derived be termed a coalescence of type $\{\mu, i\}$. The two essential features of these coalescences are their distributive nature with respect to the addition of quantities in $\mathfrak{M}$, and their associative property with respect to the operation of forming $\mathfrak{M} \cdot \bar{\mathfrak{M}}$. Both are obvious from the definitions.

Now the fact that the totality of coalescences on $\mathfrak{M}_{[1^n]}$ covers the full tensor space shows immediately, that, by the distributive property, the coalescences of two reciprocal sets $\{q_{[1^n]}\}$, $\{r_{[1^n]}\}$ cover it, and the fact that $q_{[1^n]} \cdot \bar{r}_{[1^n]} = 0$ implies

$$q_\mu^i \cdot \bar{r}_\nu^j = 0$$

for all i, j, μ and ν, shows that $\mathfrak{M}$ splits up into $\sum f_\mu$ parts, each containing a part in $\mathfrak{M}_{[1^n]}$ isomorphic with an irreducible ideal of $\mathcal{O}$, together with all its coalescences. And such that any quantity of one part annihilates the quantities of $\bar{\mathfrak{M}}$ corresponding to all those of other parts.

By/

By an obvious reciprocity in transposition the parts obtained above are mutually exclusive and one could deduce from this their orthogonality. It is, however, desirable that one should obtain a complete specification of the form of the different parts in terms of an orthogonal basis, and it is in this connection that the results of paragraph 9 become of paramount importance. Corresponding to $\mathcal{M}_\mu$ and to a part corresponding in the isomorphism to the ideal

$$\{ e_1 \bar{e}_j , e_2 \bar{e}_j \ldots\ldots\ldots e_f \bar{e}_j \}$$

of type $[\lambda]$, we shall study first what might be termed the fundamental coalescence of type $[\mu]$.

$$\mu_{ij} \longrightarrow \mu_{i1} , \quad j = 1 \ldots\ldots \lambda_i \; ; \quad i = 1 \ldots\ldots h$$

The essential character of the coalescing operation lies in the fact that it removes any distinction between symbols which go over into each other, in the isomorphism with $\mathcal{O}$, by a permutation involving only those which are coalesced. In fact it treats all those symbols similarly and sums their co-efficients as if they were identical. Now, under the fundamental coalescence considered here, the substitutions described are precisely those of the group $\mathfrak{h}_{[\mu]}$ and the corresponding operation in the group ring, performing summations over co-efficients of all substitutions equivalent on the right with respect to $\mathfrak{h}_{[\mu]}$ is that of pre-multiplying by $S_{[\mu]}$. Hence the relation between $\mathcal{M}_{[1^n]}$ and $\mathfrak{M}$ can be extended, so that $\mathfrak{M}$, together with the moduls constructed on the right co-sets of $\mathfrak{h}_{[\mu]}$ taken for each value of $[\mu]$ and repeated to cover other coalescences as described below, gives a complete representation of the properties of $\mathcal{M}$. To derive now the effects of coalescence on the basis of $\mathcal{M}_{[1^n]}$ it is sufficient to consider the products $S_{[\mu]} e_i \bar{e}_j$ discussed in detail in paragraph 8. It has been shown that these form orthogonal sets in i and j for values of i which are inequivalent and non-degenerate with respect to $[\mu]$ so that the corresponding part of $\mathcal{M}_\mu$ contains f_λ^μ orthogonal quantities, and these are each separately orthogonal to all $\mathcal{M}_\mu$ appearing in other parts of the reduction. Finally, in the extension to other $\mathcal{M}_\mu'$ it is sufficient to define the correspondences by

$$\mu_{ij} \longrightarrow K_i$$

where the K_i run through all sets of non-repeated symbols which can replace the original μ_{i1} and lead to distinct $\mathcal{M}_\mu'$, the process being valid since the actual symbols used do not affect the substitutional character of the correspondence.

The basis in $\mathcal{M}_\mu'$ is then also orthogonal and, since the $\mathcal{M}_\mu'$ do not overlap, the whole construct, suitably normalised, gives an orthogonal reduction of $A^{\{n\}}$ to the direct sum of Σf_λ matrices. To each irreducible left ideal in $\mathcal{O}$ there corresponds a unique irreducible part of $\mathcal{M}$, and to each set of isomorphic ideals of type $[\lambda]$ an irreducible matrix $a_{[\lambda]}$, it being obvious that different /

different values of j lead to the same matrix.

The reduction is illustrated on the following page by the reducing matrix for case n=3, the quantities $a_{1i_1} \ldots\ldots a_{ni_n}$ being represented by the symbols $i_1 \ldots\ldots i_n$ written above their corresponding columns. The matrix $\mathcal{A}_{[2,1]}$ is given in rational form on page 186 of Professor Littlewood's "Theory of Group Characters" and the orthogonal form is given here for the purposes of comparison.

$$
\begin{array}{c|cccccc|cccccccccccccccccc|ccc|}
 & \multicolumn{6}{c|}{M_{[1^3]}} & \multicolumn{18}{c|}{M_{[21]}} & \multicolumn{3}{c|}{M_{[3]}} \\
 & 321 & 231 & 312 & 132 & 213 & 123 & 113 & 131 & 311 & 112 & 121 & 211 & 332 & 323 & 233 & 331 & 313 & 133 & 221 & 212 & 122 & 223 & 232 & 322 & 111 & 222 & 333 \\
\hline
 & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} &&&& &&& &&& &&& &&& &&& &&& \\
 & &&& &&& \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} &&&& &&& &&& &&& &&& &&& \\
 & &&& &&& &&& \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} &&&& &&& &&& &&& &&& \\
 & &&& &&& &&& &&& \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} &&&& &&& &&& &&& \\
{[3]} & &&& &&& &&& &&& &&& \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} &&&& &&& &&& \\
 & &&& &&& &&& &&& &&& &&& \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} &&&& &&& \\
 & &&& &&& &&& &&& &&& &&& &&& \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} &&&& \\
 & &&& &&& &&& &&& &&& &&& &&& &&& 1 && \\
 & &&& &&& &&& &&& &&& &&& &&& &&& & 1 & \\
 & &&& &&& &&& &&& &&& &&& &&& &&& && 1 \\
\hline
 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} &&&&& &&& &&& &&& &&& &&& &&& \\
 & -\frac{1}{2\sqrt3} & -\frac{1}{2\sqrt3} & -\frac{1}{2\sqrt3} & -\frac{1}{2\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} &&&& &&& &&& &&& &&& &&& &&& \\
 & &&& &&& \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt6} &&&& &&& &&& &&& &&& &&& \\
{[2,1]} & &&& &&& &&& \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt6} &&&& &&& &&& &&& &&& \\
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\begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33} \\ -\tfrac{1}{2}a_{11}a_{23}a_{32} - \tfrac{1}{2}a_{13}a_{22}a_{31} \\ -\tfrac{1}{2}a_{12}a_{23}a_{31} - \tfrac{1}{2}a_{13}a_{21}a_{32} \end{matrix} &
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\end{bmatrix}
$$

The simple cases $a_{[1^n]}$ and $a_{[n]}$ can be disposed of immediately. In the former all operators $S_{[\mu]}$ give zero and hence all q_μ vanish except the generating $q_{[1^n]}$. Then by forming the single quantity $e\bar{e}$ one obtains, as described in paragraph 1, $|A| = a_{[1^n]}$. For $a_{[n]}$ all the operations produce sets of units and one obtains the corresponding normalised Schläflian.

In other cases one could proceed by reducing the invariant matrices corresponding to the ideals $(S_{[\lambda]})$. By virtue of the distributive nature of the coalescing operation, the result of any such reduction will be the direct sum of the invariant matrices generated directly, by the method described above, from the irreducible parts of $(S_{[\lambda]})$ and a complete reduction corresponding to that given for the group representation in paragraph 9 would only contribute to the theory a series of new row and column combinations representing the same irreducible form. On the other hand it is of interest to observe that the invariant matrix corresponding to $(S_{[\lambda]})$ assumes the form of a direct product of the normalised Schläflians $a_{[\lambda_1]}, a_{[\lambda_2]} \cdots\cdots a_{[\lambda_h]}$. To see this it is sufficient to partition $i_1 i_2 \cdots\cdots i_n$ to conform with $[\lambda]$ and to let the different sets range over all suffix selections independently. The complete reduction of this matrix is therefore a consequence of paragraph 9.

To describe the canonical form obtained for an irreducible invariant matrix $a_{[\lambda]}$ one can utilise the fact that the coalescing correspondence is an associative operation with respect to that of passing from m to $m \cdot \bar{m}$. In fact this shows that terms of $m_\mu \bar{m}_\nu$ can be written down directly from those of $m_{[1^n]} \cdot \bar{m}_{[1^n]}$ by coalescing the row sets in the manner appropriate to the particular m_μ^i considered, and by treating the column sets similarly in m_ν^j. But in performing this operation one has still to take account of the two normalising factors introduced in passing from $m_{[1^n]}$ to m_μ and m_ν. Since the part $m_{[1^n]} \cdot \bar{m}_{[1^n]}$ can be written down directly from the group matrices, the discussion may be regarded as complete once these factors have been evaluated, and we proceed therefore to this extension of the results of paragraph 9.

Taking $e_i \bar{e}_j$ in normalised form and writing ϑ_{ij}^2 for the sum of the squares in the corresponding group matrix form,

$$S_{[\mu]} \equiv \sum_{r,s} c_{rs}\, e_r \bar{e}_s \ (\mathcal{N})$$

implies

$$S_{[\mu]} \cdot e_i \bar{e}_j = \sum_r c_{ri}\, \vartheta_{rj}\, e_r \bar{e}_j$$

so that the normalising factor is, on account of the orthogonality,

$$\sum_r \left(c_{ri}\, \vartheta_{ij}\right)^2$$

This quantity relates to the quantities in $\mathcal{O}$ and, to pass from it to the normaliser in $\mathcal{M}_\mu$, a factor $[\mu!]$ must be suppressed since in the preceding each right co-set is repeated that number of times.

Substituting from the congruence relation in

$$\left(S_{[\mu]}\right)^2 = [\mu]!\, S_{[\mu]}$$

and taking account of the symmetry in the c_{rs} required by

$$\bar{S}_{[\mu]} = S_{[\mu]}$$

we obtain for the normaliser $c_{ii}\,\vartheta_{ii}$ after the factor $[\mu]!$ has been removed. It follows that, in any particular instance, the factor can be obtained as the scalar product of $S_{[\mu]}$ and the element of the group matrix corresponding to $e_i\bar{e}_i$

Taking account now of the results of paragraph 9, where it was shown that the c_{ij} are zero unless i, j correspond to equivalent non-degenerate lattice permutations, we obtain also the result that the non-zero $c_{ij}\,\vartheta_{ij}$ $i = 1,2\ldots\ldots$ stand in a definite ratio to each other, and in one which can be written down immediately for any permutation. The inverse ring gives a corresponding relation in j, and, combining the two, all the $c_{ij}\,\vartheta_{ij}$ can be obtained from any given one by means of these multipliers so that it would only remain to evaluate one scalar product.

We proceed instead to evaluate the sum of the normalisers $c_{ii}\,\vartheta_{ii}$ as this will prove of particular interest in the following paragraph. Let the fixed ratio be $p_1 : p_2 \ldots\ldots p_r$ and let the $c_{ii}\,\vartheta_{ii}$ corresponding to the first of these be denoted by c. Then the other $c_{ij}\,\vartheta_{ij}$, with the same value of i, are $\frac{p_j^2}{p_1^2}\cdot c$ and the $c_{jj}\,\vartheta_{jj}$ are $\frac{p_j^2}{p_1^2}\cdot c$ so that the sum of the normaliser is

$$c + \frac{p_2^2}{p_1^2}c + \frac{p_3^2}{p_1^2}c + \cdots\cdots$$

But the relation

$$\left(S_{[\mu]}\right)^2 = [\mu]!\, S_{[\mu]}$$

applied to the sets with i fixed, gives

$$c^2 + \frac{p_2^2}{p_1^2}c^2 + \frac{p_3^2}{p_1^2}c^2 + \cdots\cdots = [\mu]!\cdot c$$

so that the sum of the normalisers is simply $[\mu]!$. One could evaluate from this result the various independent normalisers and so construct the whole invariant matrix in canonical form by studying the lattice permutations above, and without reference to the scalar products.

11. FUNDAMENTAL IDEAL INVARIANTS AND SYMMETRIC FUNCTIONS.

Up to this point no reference has been made to the special properties of the ideal invariants

$$I_{[\lambda]} = \sum e_i \bar{e}_i \quad . \quad e_i \in \mathfrak{A}_{[\lambda]}$$

invariant under a change of e_i basis and under the change from $\{e_1 \ldots\ldots e_s\}$ to any isomorphic ideal. These correspond to the immanants of Professor Littlewood's "Theory of Group Characters" if one sets up an obvious isomorphism between parts of $\mathcal{O}$ and $\mathfrak{M}_{[1^n]} \cdot \bar{\mathfrak{M}}_{[1^n]}$. The correspondence is however, in one sense at least, improper and will be dismissed in favour of a much more fruitful one developed in this section between the centrum and quantities ranging over all $\mathfrak{M}_\mu \cdot \bar{\mathfrak{M}}_\mu$.

The invariance of the quantities $I_{[\lambda]}$ involves in particular

$$\tau \cdot I_{[\lambda]} \cdot \tau^{-1} = I_{[\lambda]}$$

showing that $I_{[\lambda]}$ is in the centrum of $\mathcal{O}$. Writing

$$I = \sum_\tau \chi(\tau) \cdot \tau \quad , \quad \chi(\sigma\tau\sigma^{-1}) = \chi(\tau),$$

$\chi(\tau)$ is the character of τ in the representation considered. All the orthogonal properties of the characters are specialisations of results derived in earlier sections. Ideal invariants are added in direct sum.

The conjugate isomorphism of $\mathcal{O}$ does not have a direct analogue in tensor space since quantities derived by coalescence in one ring can be replaced by zeros in the other. For example, there is at first sight little to suggest a relation between $\mathcal{A}_{[n]}$ and $\mathcal{A}_{[1^n]} = |A|$. The symmetry can however be restored, at least as far as the traces of invariant matrices are concerned, by means of a correspondence which will now be developed.

Let quantities

$$a^{ij} = \left(a_{i1}\alpha_{j1} + a_{i2}\alpha_{j2} + \ldots\ldots\ldots + a_{in}\alpha_{jn} \right)$$

be defined for products of m such symbols, in which the i, j both cover a permutation of $1, 2, \ldots\ldots m$ so that the expanded forms can be interpreted by means of the distributive law, and consider the product

$$a^{1i_1} a^{2i_2} \ldots\ldots\ldots a^{ni_n}$$

with $i_1 i_2 \cdots i_n$ a permutation of $1, 2 \cdots\cdots n$. The multinomial theorem shows immediately that, expanded and written in terms of the a_{ij}, this product is

$$n!\ a_{i_1 1} a_{i_2 2} \cdots\cdots a_{i_n n}$$

together with $\binom{n}{\mu}$ times all the coalescences of type $[\mu]$ obtained from this in the manner used in defining the correspondence between $m_{[1^n]}$ and $m_{[\mu]}$. Now, considering diagonal terms, the process of passing directly from $m_{[1^n]} \cdot \bar{m}_{[1^n]}$ to $m_{[\mu]} \cdot \bar{m}_{[\nu]}$ produces, for any fixed $e_i \cdot \bar{e}_i$, the required part of the trace increased by a factor which is the corresponding normaliser, taken without the radical. Moreover, direct coalescences from other $e_i \bar{e}_i$ equivalent with respect to $[\mu]$ produce corresponding multiples of the particular part considered, so that, by the result obtained above, this part appears in excess by a factor $[\mu]!$ Hence under the reciprocal correspondence

$$\begin{pmatrix} 1 & 2 & \cdots\cdots & n \\ i_1 & i_2 & \cdots\cdots & i_n \end{pmatrix} \longleftrightarrow \frac{1}{n!}\, a^{1 i_1} \cdots\cdots a^{n i_n},$$

$$I_{[\lambda]} \longleftrightarrow \mathcal{J}_{[\lambda]}$$

where $\mathcal{J}_{[\lambda]}$ is the trace of $a_{[\lambda]}$. For the second relation one can pass freely to the inverse ring since the conjugate classes are self-inverse.

Now, it has been shown in Professor Turnbull's paper quoted in the introduction, and can be seen immediately from the form of A^r that the product

$$a^{12} a^{23} \cdots\cdots a^{r1}$$

represents the sum of the r^{th} powers of the latent roots of A. In this the symbols $1 \cdots\cdots r$ can be replaced by any equivalent set provided that they form a cycle closing at the r^{th} term.

Expressing $\begin{pmatrix} 1 & 2 & \cdots\cdots & n \\ i_1 & i_2 & \cdots\cdots & i_n \end{pmatrix}$ in the cycle notation, it appears that $\mathcal{J}_{[\lambda]}$ is a symmetric function of degree n of the latent roots of A. To show that this is actually the bi-alternant $h_{\{\lambda\}}$ we consider the I corresponding to $(S_{[\lambda]})$. Since it consists of all the transforms of $S_{[\lambda]}$ it corresponds to products of permanents $\overset{+}{|} a^{ij} \overset{+}{|}$ of orders $\lambda_1, \lambda_2 \cdots\cdots$

In extended form, and written in terms of the cycle types any such permanent gives an isobaric series in the products of sums of powers, with co-efficients enumerating the elements in each cycle type, so that it represents $n!$ times the corresponding complete homogeneous function of the latent roots. Hence the $\mathcal{J}$ corresponding to $(S_{[\lambda]})$ is the product $h_{\lambda_1} h_{\lambda_2} \cdots\cdots h_{\lambda_h}$ of complete homogeneous symmetric functions. Deriving the $a_{[\lambda]}$ successively from the $(S_{[\lambda]})$ by removing parts of lower order, a theorem of Professor Littlewood's on the co-efficient of the h-bi-alternant $h_{\{\mu\}}$ in the expansion of a product $h_{\lambda_1} \cdots\cdots h_{\lambda_h}$ shows immediately by the results of paragraph 9, that the symmetric function $\mathcal{J}_{[\lambda]}$ is indeed an h-bi-alternant. Frobenius' theorem on the products of sums of powers is implied. Also, as shown in Professor Turnbull's paper, $|a^{ij}|$ represents the corresponding elementary symmetric function. This is otherwise obvious from a well known result in the theory of symmetric functions.

Finally $I^*_{[\lambda]} = I_{[\lambda^*]}$ expresses a corresponding relation for /

for $\mathcal{J}^*_{[\lambda]}$, where the conjugate transformation applies to the symbolic form and converts permanents into determanants and products of h_{λ_i} into products of elementary symmetric functions.

2. Reviewing the results of the discussion, it seems that some extensions of reasonable interest have been contributed. The nature of these can be described as an extension of well-known results concerning traces of matrices to properties of the matrices as a whole, so that the trace relations then appear as particular-:isations of the more general discussion. From the more abstract point of view the chief interest seems to be in the transference of results in the group ring to an analogous set of results in tensor space, by means of a particularly simple correspondence.

The results of paragraphs 1 to 4 consist essentially of an adaptation of the Frobenius theory of group representation to the particular case of orthogonal reduction. The greater part of the discussion is not restricted to the symmetric group and may be applied immediately to the orthogonal representations of any finite group, subject only to the assumption of complete reducibility in K.

The direct construction of paragraphs 5 and 6 seems to achieve as simple a definition of the orthogonal group matrices as one would expect to obtain, and, at least, one sufficient for the later applications and for the familiar results concerning the group characters. The idea of using the ideals $(S_{[\lambda]})$ as "filters" to sift out the various representations was suggested by the corres-:ponding manner in which bi-alternants can be defined in terms of products of complete homogeneous symmetric functions. Taking paragraph 9 into account, it seems that one is justified in preferring the present procedure to that followed by Young. It was felt that the asymmetry introduced by selecting a basis of standard tableaux did not give a sufficiently natural approach to the problem, and the theory was therefore constructed quite independently It is, however, natural that the two constructions should yield isomorphic ideal bases, and therefore identical representations, since they both proceed by an induction on n and so involve a direct sum of earlier representations. Then, by selecting the elements of the matrix algebras, or rather their analogues in the ring, as a basis for $\mathcal{O}$, the two sets naturally become identical.

It might be felt that some explanation is necessary for the manner in which new notations have been introduced, in some cases replacing conventional forms. But, as remarked above, most of the attention given to the subject has been focussed on the trace relations and there have been comparatively few presentations involving quantities of the type considered here. An attempt has therefore been made to develop a notation which would meet all the requirements of the isomorphisms considered. In particular, some of the results could have been expressed in a more familiar form in terms of Young tableaux. The standard tableaux, regarded merely as /

as enumerants, have been replaced by their equivalent lattice permutations, as these seem to arise more naturally in the reductions.

The results of paragraph 9, which lead directly to the reduction of tensor space, are suggested by corresponding theorems developed by Professor Littlewood for the expansions of products of complete homogeneous symmetric functions. Here again, the discussion forms an extension from trace relations to properties of complete matrices. Although the form of the results could have been deduced fromthe $\mathfrak{M} \rightarrow \mathfrak{M}_{[1^n]}$ correspondence and the trace relations, the full reduction of the invariant matrices and the more complete correspondence involving the whole modul $\mathfrak{M}$ requires the matric result for its specification.

Finally, it has been considered sufficient to present the results of paragraph 11 in abbreviated form. The trace relations themselves as distinct from the isomorphism have received such exhaustive treatment that it is not surprising that the development does not lead directly to new results. It is, however, interesting in that it reverses the usual form of argument and develops the theory entirely from the properties of the group ring. For the sake of completeness the use of known relations concerning traces has been deliberately avoided in earlier sections. In this respect the investigation follows the lines suggested by Professor Weyl ("The Classical Groups"), although it might be pointed out that the construction of a one-sided ideal basis is in no wise more arbitrary than the two-sided construction.

It is hardly necessary to point out that the standpoint of modern algebra has been adopted throughout because of the great flexibility which it allows in moving from one type of correspondence to another, and in relating systems which would otherwise have been more closely characterised by the particular type of notation in which they were expressed and might therefore have seemed to have nothing in common.

Since from this general point of view many standard results fall naturally into the development, it does not seem necessary to refer to them as isolated theorems when applying them to the discussion. It will, for example, be found that Schur's Lemma appears in various forms in the course of the argument. It might also be mentioned here that, although the presentation of the argument deriving the properties of invariant matrices from the group ring has been made as complete as seemed desirable, no attempt has been made to follow up consequences leading to standard theorems, unless these have had a special bearing on the subsequent discussion.

In developing the results along these lines frequent reference has been made to Frobenius' original papers (Berlin Sitzungsberichte 1894-1906). The greatest stimulus has, however, been provided by a course of research lectures delivered by Professor A.C. Aitken, and his illuminating account of the theory has proved an ever fruitful source of ideas and an unfailing guide at every mile stone.

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